

Objectives:

- Define definite integrals.
- Find areas under curves using definite integrals.

Definitions: If f is a function defined for $a \leq x \leq b$, we divided the interval $[a, b]$ into n subintervals of equal width

$$\Delta x = \frac{b - a}{n}.$$

We let $x_0 = a, x_1, \dots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals, so x_i^* is in the i th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=0}^n f(x_i^*) \Delta x$$

provided the limit exists. If the limit does exist, we say that f is integrable on $[a, b]$.

Terminology: Let's break down the notation $\int_a^b f(x) dx$.

- The symbol \int is called an integral sign
- $f(x)$ is the integrand
- a and b are the limits of integration
- a is the lower limit of integration and b is the upper limit of integration
- We call computing an integral integration

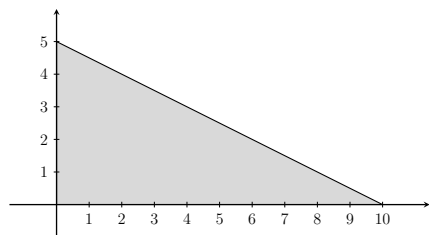
Some intuition: The definite integral is computing area between the curve and the x -axis but we consider any area above the x -axis is positive and any area underneath the x -axis is negative.

But wait! Our definition shows that the definite integral is also the limit of Riemann sums!

Some useful things:

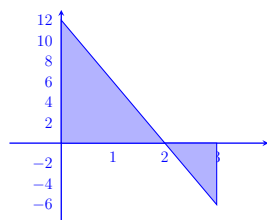
- The sum of the integers from 1 to n : $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
- The sum of the squares of integers from 1 to n : $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
- The sum of the cubes of integers from 1 to n : $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$

Example 1 Write down a definite integral that gives the area of the shaded region.



$$\int_0^{10} -\frac{1}{2}x + 5$$

Example 2 Evaluate $\int_0^3 12 - 6t \, dt$ by drawing a the region and computing the area.



$$\text{Area} = \frac{1}{2}(2)(12) - \frac{1}{2}(1)(6) = 12 - 3 = 9$$

Example 3 Evaluate $\int_0^2 \sqrt{4 - x^2} \, dx$ by drawing a the region and computing the area.

$y = \sqrt{4 - x^2}$ is the upper half of the circle $x^2 + y^2 = 4$, which has center $(0, 0)$ and radius 2. Taking the integral from 0 to 2 gives the area of the right half of this semicircle. $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(2)^2 = \pi$.

Example 4 A table of values of $f(x)$ is given below. Estimate $\int_0^{12} f(x) \, dx$ using Riemann sums.

| | | | | | |
|--------|----|----|----|----|----|
| x | 0 | 3 | 6 | 9 | 12 |
| $f(x)$ | 32 | 22 | 15 | 11 | 9 |

Right Riemann sum with $n = 4$:

$$3 \cdot 22 + 3 \cdot 15 + 3 \cdot 11 + 3 \cdot 9 = 171.$$

Left Riemann sum with $n = 4$:

$$3 \cdot 32 + 3 \cdot 22 + 3 \cdot 15 + 3 \cdot 11 = 240.$$

Example 5 Calculate $\int_0^2 x^3 dx$ exactly using a limit of Riemann sums. We can do this computation in either summation notation or the expanded form.

The right Riemann sum set up: n rectangles; $\Delta x = \frac{2}{n}$; right endpoints: $\frac{2}{n}, \frac{4}{n}, \dots, \frac{2n}{n}$; heights: $(\frac{2}{n})^3, \dots, (\frac{2n}{n})^3$; summation: $\sum_{i=1}^n (\frac{2i}{n})^3 \cdot \frac{2}{n}$. Putting all of this together, we can compute our integral:

$$\begin{aligned}
 \int_0^2 x^3 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n}\right)^3 \cdot \frac{2}{n} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2^3 i^3}{n^3} \cdot \frac{2}{n} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2^4}{n^4} i^3 \\
 &= \lim_{n \rightarrow \infty} \frac{2^4}{n^4} \sum_{i=1}^n i^3 \\
 &= \lim_{n \rightarrow \infty} \frac{2^4}{n^4} \left(\frac{n^2(n+1)^2}{4}\right) \\
 &= \lim_{n \rightarrow \infty} \frac{2^2}{n^2} (n+1)^2 \\
 &= \lim_{n \rightarrow \infty} 4 \frac{n^2 + 2n + 1}{n^2} \\
 &= \lim_{n \rightarrow \infty} 4 \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \\
 &= 4.
 \end{aligned}$$

So the area under the curve x^3 between $x = 0$ and $x = 1$ is exactly 4. Cool!

Theorem If $f(x)$ is continuous on $[a, b]$, or if $f(x)$ has only a finite number of jump discontinuities, then f is integrable on $[a, b]$, i.e., the definite integral $\int_a^b f(x) dx$ exists.

Things to note: We have assumed that $a < b$ for defining $\int_a^b f(x) dx$, but the Riemann sum will allow $a > b$. If $a > b$, then Δx used to be $\frac{b-a}{n}$ and is now $\underline{\Delta x = \frac{a-b}{n}}$. So we have

$$\int_b^a f(x) dx = - \int_a^b f(x) dx$$

What if $a = b$? Then $\Delta x = \underline{\frac{a-a}{n} = 0}$ so

$$\int_a^a f(x) dx = 0$$

Properties of Definite Integrals: Let $f(x)$ and $g(x)$ be continuous functions and c some constant number.

1. $\int_a^b c dx = c(b-a)$
2. $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$
3. $\int_a^b cf(x) dx = c \int_a^b f(x) dx$
4. $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$
5. $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$

Example 6 Evaluate $\int_0^2 (4 + 5x^3) dx$.

$$\begin{aligned} \int_0^2 (4 + 5x^3) dx &= \int_0^2 4 dx + \int_0^2 5x^3 dx \\ &= 4 \cdot 2 + 5 \int_0^2 x^3 dx \\ &= 4 \cdot 2 + 5 \cdot 4 \\ &= 28. \end{aligned}$$