Objectives:

- Find a function that models a problem and apply the techniques from 4.1, 4.2, and 4.3 the find the optimal or "best" value.

Suggested procedure:

Step 1. Draw a picture! Label variables and known quantities.

Step 2. Decide what quantity we want to maximize (or minimize).

Step 3. Find a formula for the quantity that we want to maximize (or minimize).

Step 4. Use constraints to turn our formula into an equation in one variable.

Step 5. Find the domain.

Step 6. Find the global minimum (or maximum).

(a) If we are looking in a closed interval: substitute endpoints and critical points into the function and choose the largest (or smallest) value.

(b) If we are looking in an open interval: Hope there is only one critical point, show there is a local maximum (or minimum) there, conclude it is also a global maximum (or minimum).

Step 7. Remember to answer the original question clearly and completely!

Example 1. Find two non-negative numbers whose sum is 200 and whose product is maximum.

Step 1. Let x and y be the two numbers. Then $x + y = 200$, so $y = 200 - x$. (If we want a picture we can draw a rectangle with sides x and $y = 200 - x$.)

Step 2. We want to maximize the product of x and y, (the area in the rectangle).

Step 3. $P = xy$

Step 4. $p(x) = x(200 - x) = 200x - x^2$

Step 5. The domain of $p(x)$ is $(0, 200)$ since x must be positive and $y = 200 - x$ must be positive.

Step 6. To find x so that this product is maximized, first find the derivative of $p(x)$:

$$p'(x) = 200 - 2x$$

Then we set the derivative equal to 0 and solve for x, with the result $x = 100$ To check if this critical point, $x = 100$ is a maximum, we have two options:

(a) **First Derivative Test:** Construct a numberline for $p'(x)$. This will show $p'(x)$ changes from positive to negative at $x = 100$, so there is a local maximum at $x = 100$. Since $p'(x)$ only changes sign once, $p(100)$ is an absolute maximum.

(b) **Second Derivative Test:** $p''(x) = -2$, so $p(x)$ is always concave down. Thus, $p(100)$ a local maximum AND since there is only one critical point, a global maximum.

Step 7. So, $x = 100$ and $y = 200 - 100 = 100$. The two numbers are 100 and 100.

Example 2. The corners are cut our of an $8\frac{1}{2}'' \times 11''$ piece of paper and it is folded into a box. What size squares should be removed to maximize the volume?

Step 1. Draw a diagram. Suppose we label the side length of each removed square as x.

Step 2. We want to maximize the volume.

Step 3. Volume is given by

$$V(x) = (\text{length})(\text{width})(\text{height})$$

Step 4.

$$V = (11 - 2x)(8.5 - 2x)(x) = (93.5 - 39x + 4x^2)x = 93.5x - 39x^2 + 4x^3$$

Step 5. The length, width, and height all need to be positive, so we need $x > 0$, $11 > 2x$, $8.5 > 2x$. So x is greater than 0 and less than $8.5/2 = 4.25$. The domain is $(0, 4.25)$.

Step 6. We can find the critical points of $V(x)$ by taking the derivative and setting the resulting derivative equal to 0:

$$\begin{aligned} V'(x) &= 93.5 - 78x + 12x^2 \\ 0 &= 93.5 - 78x + 12x^2 \\ x &= \frac{78 \pm \sqrt{78^2 - 4(12)(93.5)}}{2(12)} \quad \text{(Quadratic Equation)} \\ x &\approx 1.585, 4.915 \end{aligned}$$

The only critical point in our domain is $x \approx 1.585$, so if we can show $V(1.585)$ is a local maximum, it will also be an absolute maximum. Use the first or second derivative test to show $V(1.585)$ is a local maximum.

Step 7. The volume will be maximized if the squares removed have side length $\dfrac{78 - \sqrt{78^2 - 4(12)(93.5)}}{2(12)} \approx$ 1.585 inches

Example 3. A rectangle is inscribed in the triangle with vertices $(0,0)$, $(4,0)$, and $(0,8)$ with one side of the rectangle on lying on the x-axis and one side of the rectangle lying on the y-axis. What is the maximum area of the rectangle?

Step 1. Draw the triangle and an example rectangle inside of the triangle. Label the sides of the rectangle. In this solution we call the width x and the height y. The point of the rectangle on the triangle's hypotenuse has coordinates (x,y).

Step 2. We want to maximize the area of the rectangle.

Step 3. Area of the rectangle is given by $A = xy$.

Step 4. To write y in terms of x, we use the fact that (x,y) is on the hypotenuse of the triangle. If the hypotenuse is extended to be a line, the equation for that line is $y - 8 = \frac{8-0}{0-4}(x-0)$, or $y = -2x + 8$. So,

$$A(x) = xy = x(-2x+8) = -2x^2 + 8x$$

Step 5. The domain is $(0,4)$ since the width of the rectangle must be between 0 and 4.

Step 6. $A'(x) = -4x + 8$, so the only critical point is $x = 2$. Use the first or second derivative test to show there is a local maximum at $x = 2$ and then justify why there is also an absolute maximum at $x = 2$.

Step 7. The area is maximized when $x = 2$. The question asks what is the maximum area, so we should compute what the area is when $x = 2$. $A(2) = -2(2)^2 + 8(2) = 8$, so the maximum area is 8 square units.

Example 4. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Step 1. Draw a graph of $y^2 = 2x$ and label $(1, 4)$

Step 2. Let (a, b) be a point on the parabola. We want to minimize the distance from (a, b) to $(1, 4)$.

Step 3. The distance between (a, b) and $(1, 4)$ is given by:

$$D = \sqrt{(a-1)^2 + (b-4)^2}$$

Step 4. Since (a, b) is on the parabola $y^2 = 2x$, we know $b^2 = 2a$ and so $a = \dfrac{b^2}{2}$.

$$D(b) = \sqrt{\left(\frac{b^2}{2} - 1\right)^2 + (b-4)^2}$$

Step 5. b can be any real number, so the domain of $D(b)$ is $(-\infty, \infty)$.

Step 6. $D'(b) = \frac{1}{2}\left(\left(\frac{b^2}{2} - 1\right)^2 + (b-4)^2\right)^{-1/2}\left(2\left(\frac{b^2}{2} - 1\right)(b) + 2(b-4)\right) = \dfrac{b^3 - 8}{2\sqrt{\left(\frac{b^2}{2} - 1\right)^2 + (b-4)^2}}$.

$D'(b) = 0$ only when $b^3 - 8 = 0$. So, the only critical point is $b = 2$.
MAKE SURE to use the first or second derivative test to show there is a local minimum at $b = 2$ AND justify that this is also an absolute minimum.

Step 7. The point (a, b) on the parabola $y^2 = 2x$ with minimum distance from $(1, 4)$ is where $b = 2$, so $a = \dfrac{2^2}{2} = 2$. So the point is $(2, 2)$. (It's a good idea to draw this point on your graph from Step 1. to check that this is reasonable.)

Example 5. A rectangular mural will have a total area of 24 ft^2 which includes a border of 1 ft on the left, right, and bottom and a border of 2 ft on the top. What dimensions maximize the total paintable area inside the borders.

Step 1. Draw a rectangle contained in another rectangle. In this solution we call the width of the outer rectangle x and the height of the outer rectangle y. Then label the width of the inner rectangle as $x-2$ and the height of the inner rectangle as $y-3$.

Step 2. We want to maximize the area of the inner rectangle.

Step 3. The area of the inner rectangle is given by $A = (x-2)(y-3)$

Step 4. We know that $xy = 24$. So, $y = \frac{24}{x}$. Then we have the equation $A(x) = (x-2)\left(\frac{24}{x} - 3\right)$

Step 5. The inner rectangle dimensions must be positive. (If they are positive, the outside dimensions will be positive too.) So, we need $x - 2 > 0$ and $\frac{24}{x} - 3 > 0$. Solving these constraints we get $x > 2$ and $x < 8$. So the domain is $(2, 8)$.

Step 6. $A'(x) = -3 + \frac{48}{x^2}$ so we solve for the critical points $x = \pm\sqrt{\frac{48}{3}} = \pm 4$. There is only one critical point in our domain: $x = 4$. So, use the 1st or 2nd derivative test to show there is a local maximum at $x = 4\sqrt{3}$ and then justify that this is also a global maximum.

Step 7. The inner area is maximized when $x = 4$, so the outer dimensions that maximize area are 4 feet wide by $\frac{24}{4} = 6$ feet high. The corresponding inner dimensions are 2 feet wide by 3 feet high.

Example 6. A can is made to hold 1 liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.
(Note: 1 liter is equivalent to 1,000 cubic centimeters.)

Step 1. Draw a can. This is a cylinder, so we label the dimensions: radius (r) and height (h).

Step 2. The cost of metal to manufacture the can depends on the **surface area** of the can. So, we want to minimize the surface area of the cylinder.

Step 3. Surface area is given by $SA = 2\pi r^2 + 2\pi rh$. (Note that this is the sum of two circle areas and the area of a rectangle with height h and width $2\pi r$, the circumference of the circle. Interestingly, we can also derive this formula by differentiating $V = \pi r^2 h$ and letting $r' = h' = 1$.)

Step 4. To rewrite h in terms of r, we will use the fact that the volume of the can is 1,000 cubic centimeters. So if h and r are in cm, $\pi r^2 h = 1,000$ so $h = \dfrac{1,000}{\pi r^2}$. So now we are trying to minimize surface area, given as a function of r by:

$$SA(r) = 2\pi r^2 + 2\pi r\left(\frac{1,000}{\pi r^2}\right) = 2\pi r^2 + \frac{2,000}{r}$$

Step 5. The radius must be positive, so the domain is $(0, \infty)$.

Step 6. $SA'(r) = 4\pi r - \dfrac{2,000}{r^2}$. To find critical points:

$$\begin{aligned} 0 &= 4\pi r - \frac{2,000}{r^2} \\ 0 &= 4\pi r^3 - 2,000 \\ 2000 &= 4\pi r^3 \\ \frac{2,000}{4\pi} &= r^3 \\ \sqrt[3]{\frac{500}{\pi}} &= r \end{aligned}$$

Then be sure to use the 1st or 2nd derivative test to show this critical point is a local minimum and then justify why it is a global minimum.

Step 7. The cost of metal for the can is minimized when $r = \sqrt[3]{\dfrac{500}{\pi}} \approx 5.42$ cm, so the height is $\dfrac{1,000}{\pi \sqrt[3]{\frac{500}{\pi}}^2} = 2\sqrt[3]{\dfrac{500}{\pi}} \approx 10.84$ cm.

Example 7. A glass fish tank is to be constructed to hold 72 ft^3 of water. The top is to be open. The width will be 5 ft but the length and the depth are variable. Building the tank costs \$10 per square foot for the base and \$5 per square foot for the sides. What is the cost of the least expensive tank?

Step 1. Draw a rectangular prism. The width should be labeled 5 ft. In this solution we will use the label ℓ for length and d for depth.

Step 2. We want to minimize the cost of the tank. In this case, this is more complicated than minimizing the surface area, since not all parts of the surface cost the same amount.

Step 3. First we find the cost for the sides. Two sides will be $5d$ square feet and two sides will be ℓd square feet, so the total area for the sides is $10d + 2\ell d \text{ft}^2$. The cost for the sides is \$5 per square foot, so the cost to build the sides is $\left(10d + 2\ell d \text{ft}^2\right)\left(5\frac{\$}{\text{ft}}\right) = 50d + 10\ell d$ dollars.

Next we find the cost for the base: $(5\ell \text{ft})\left(10\frac{\$}{\text{ft}}\right) = 50\ell$ dollars.

So the total cost of the tank in dollars is given by $C = 50d + 10\ell d + 50\ell$

Step 4. We know the volume of the tank is 72 ft^3, so $72 = 5\ell d$ and thus $\ell = \frac{72}{5d}$. So

$C(d) = 50d + 10\frac{72}{5d}d + 50\frac{72}{5d} = 50d + 144 + \frac{720}{d}$.

Step 5. All of the side lengths must be positive, so $d > 0$ and $\frac{72}{5d} > 0$. We know $\frac{72}{5d} > 0$ whenever $d > 0$, so our domain is $(0, \infty)$.

Step 6. Our critical points are $\pm\sqrt{14.4}$ so the only critical point in the domain is $\sqrt{14.4} \approx 3.79$. Be sure to check if this is a local max or min and justify if it is a global extremum.

Step 7. We have found the depth d that minimizes tank costs, but the question asks "What is the cost of the least expensive tank?" so we need to find the cost of the tank when $d = \sqrt{14.4}$. $C(\sqrt{14.4}) = 50\sqrt{1.4} + 144 + \frac{720}{\sqrt{14.4}} = 100\sqrt{14.4} + 144 \approx 523.47$. So our final answer is "The cost of the least expensive tank is \$ 523.47.

Example 8. A baseball team plays in a stadium that holds $55,000$ spectators. with ticket prices at \$10, the average attendance has been $27,000$. Some financial experts estimated that prices should be determined by the function $p(x) = 19 - \frac{1}{3000}x$ where x is the number of tickets sold. What should the price per ticket be to maximize revenue?

Step 1. There's no picture to be drawn for this one.

Step 2. We want to maximize revenue, the money coming in from ticket sales.

Step 3. The revenue is given by multiplying the cost per ticket by the number of tickets. So, $R = px$ where p is the price per ticket and x is the number of tickets sold.

Step 4. We're given an equation for what price to charge if x people buy tickets, $p(x) = 19 - \dfrac{1}{3000}x$. So, $R(x) = \left(19 - \dfrac{1}{3000}x\right)x = 19x - \dfrac{1}{3000}x^2$.

Step 5. No more than 55,000 tickets can be sold. 0 tickets could be sold, but it doesn't make sense to sell a negative number of tickets. So the domain for x is $[0, 55000]$.

Step 6. Since we have a continuous, differentiable function (it's a polynomial!) on a **closed interval**, there must be an absolute minimum, and it must be at an endpoint or critical point. We already know the endpoints: $x = 0, 55000$, so let's find the critical points.
$R'(x) = 19 - \dfrac{1}{1500}x$ so the only critical point is $x = 28,500$. Next, we compare values of $R(x)$ at critical points and endpoints:

x	$R(x)$
0	0
28,500	270,750
55,000	36,666.67

So, we can conclude that the absolute maximum is \$270,750 and the max occurs at 28,500 tickets sold.

Step 7. The question asks what price should be charged per ticket, so we still need to find the price charged per ticket when 28,500 tickets are sold. $p(28500) = 9.5$. So \$9.50 should be charged per ticket to maximize revenue.