Objectives:

- Practice deciding when to use l'Hospital's rule.
- Use l'Hospital's rule together with logarithmic differentiation to find limits that have exponential indeterminate forms.

List of Indeterminate Forms:

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 1^{\infty}, \infty^0, 0^0$

To convince ourselves of the exponential forms, we can wildly abuse some notation and use logarithms, noting that $\lim_{x\to 0^+} \ln(x) = -\infty, \lim_{x\to\infty} \ln(x) = \infty$. For example, if y looks like 0^0 then $\ln(y)$ looks like $0 \cdot \ln(0) = 0 \cdot -\infty$, which is indeterminate.

If y looks like 0^{∞}, then $\ln(y)$ looks like $\infty \cdot \ln(0)$ "=" $\infty \cdot -\infty$ "=" $-\infty$, which is NOT indeterminate.

Remember: We can only use l'Hospital's rule to find $\lim\limits_{x\to\infty} f(x)$ if this limit has

indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

Even if the limit has this form, l'Hospital's rule may not be the only way to find the limit.

If the limit does not have form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can try to rewrite the function so it does have one of these forms and then apply l'Hospital's rule.

Examples:

1. $\lim\limits_{x\to 0^+} x\ln(x)$

   Form: $0 \cdot -\infty$ $\lim\limits_{x\to 0^+} x\ln(x) = \lim\limits_{x\to 0^+} \dfrac{\ln(x)}{\frac{1}{x}}$ New form: $\dfrac{-\infty}{\infty}$

   $$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x}\frac{-x^2}{1} = \lim_{x\to 0^+} -x = 0$$

2. $\lim\limits_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

   Form: 1^{∞}. Let $L = \lim\limits_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$. Then $\ln(L) = \lim\limits_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right)$. Form of $\ln(L)$ is $0 \cdot \infty$. Keep rearranging...

   $\ln(L) = \lim\limits_{x\to\infty} \dfrac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$ has form $\frac{0}{0}$ so at last we can apply l'Hôpital.

   $\ln(L) = \lim\limits_{x\to\infty} \dfrac{\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{1}{1+\frac{1}{x}} = \dfrac{1}{1+0} = 1$. Since $\ln(L) = 1$, $L = e^1 = e$.

Practice with l'Hôpital's Rule: For each limit:

(a) Write the form of the limit AND state whether the form is indeterminate.

(b) Find the limit or show that the limit does not exist.
If you use l'Hôpital's rule, state why you can use the rule.

1. $\lim_{x\to 0} \frac{(\sin(x))^2}{x}$

   Form: $\frac{0}{0}$ so can use l'H. $\lim_{x\to 0} \frac{(\sin(x))^2}{x} = \lim_{x\to 0} \frac{2\sin(x)\cos(x)^2}{1} = \frac{0}{1}$

2. $\lim_{x\to \frac{\pi}{2}} \frac{\sin(x)-1}{x}$

   Form: $\frac{0}{\pi/2}$. Not indeterminate. $\lim_{x\to \frac{\pi}{2}} \frac{\sin(x)-1}{x} = \frac{0}{\pi/2} = 0$

3. $\lim_{x\to 0} \frac{2^x - 1}{x}$

   Form: $\frac{0}{0}$ so can use l'H. $\lim_{x\to 0} \frac{2^x-1}{x} = \lim_{x\to 0} \frac{\ln(2)2^x}{1} = \frac{\ln(2)2^0}{1} = \ln(2)$

4. $\lim_{x\to\infty} \frac{x^2}{e^x}$

   Form: $\frac{\infty}{\infty}$ so use l'H. $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$ still $\frac{\infty}{\infty}$ form so use l'H again. $\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

5. $\lim_{x\to 0^+} \sqrt{x}\ln(x)$

   Form: $0\cdot -\infty$ so write $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$, now form is $\frac{-\infty}{\infty}$, so use l'H to get $\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = 0$

6. $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{3x}$=L

   Form: 1^∞, write $\ln(L) = \lim_{x\to\infty} 3x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{3x}}$, now form is $\frac{0}{0}$, so use l'H to get

   $\ln(L) = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\left(\frac{-2}{x^2}\right)}{\frac{-1}{3x^2}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}(-2)}{\frac{-1}{3}} = (-3)\frac{1}{1+0}(-2) = 6$, so $L = e^6 \approx 403$

7. $\lim_{x\to 0^+} x^{\sin(x)}$=L

   Form 0^0, write $\ln(L) = \lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)}$. New form is $\frac{\infty}{\infty}$ so use l'H to get $\ln(L) = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc(x)\cot(x)} = \lim_{x\to 0^+} \frac{-(\sin(x))^2}{x\cos(x)}$. Now form is $\frac{0}{0}$, use l'H to get $\ln(L) = \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{\cos(x)-x\sin(x)} = \frac{0}{1} = 0$. Then $L = e^0 = 1$.

8. $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}}$

   Form: $\frac{\infty}{\infty}$ so use l'H to get: $\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-3/2}} = \lim_{x\to\infty} 3x^{2/3}(x^{-1}) = \lim_{x\to\infty} 3x^{-1/3} = 0$