Objectives:

- Apply knowledge of derivatives to rates of change in many fields.
- Practice challenging derivation techniques.

Note: These applications are only a few possible applications of the derivative. Be prepared to apply your general knowledge of derivatives to many different real-world applications.

Physics Application: Motion of an Object

Let $s(t)$ be a function that gives the position of a moving object at time t.

- The function that represents the velocity is __________ .
- The function that represents the acceleration is __________ .
- To find when an object is moving forward/upward, find intervals where __________ .
- To find when an object is moving backward/downward, find intervals where __________ .
- To find when an object is stationary, look for points where __________ .
- Suppose $s(t)$ represents the height of a falling object.
  - To find the time when maximum height is achieved, we should
    (1) ______________________________
    (2) ______________________________
    (3) ______________________________
    (4) ______________________________
  - To find the time of impact, we should ____________________
  - To find impact velocity, we should ____________________
- The object is speeding up when ____________________ .
- The object is slowing down when ____________________ .

Physics Application:

A pumpkin spice muffin is launched upward from the roof of a building that is 64 feet high. The muffin is thrown with an initial velocity of 48 ft/sec. Its height in feet after t seconds is given by $h(t) = -16t^2 + 48t + 64$.

(a) On what interval(s) is the muffin moving upward?

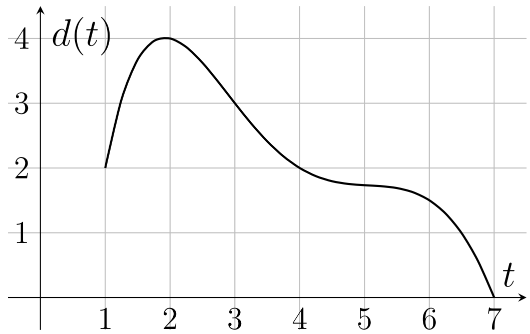
(b) When does the muffin reach maximum height? What is the maximum height?

(c) When does the muffin hit the ground? How fast is it going when it hits the ground?

(d) On what interval(s) is the muffin speeding up?

Graphical problem:

Below is a graph of the position of an object in centimeters relative to some fixed point at t seconds.



Answer the following questions. Explain how you arrived at your answer.

(a) On what interval(s) is the object slowing down?

(b) At what time(s) is the object stationary?

(c) On what interval(s) is the object moving away from the fixed point?

(d) At what time does the object reach its maximum speed?

Modeling Animal Populations:

The population of elk in a national park is modeled by $P(t) = \dfrac{2500}{1 + 24e^{-5t}}$, where t is time in decades since 1940.

(a) At $t = 0$, what is the population of the elk?

(b) Calculate $P'(0)$. Give an interpretation of this quantity in the context of this problem. Remember to use units in your answer.

(c) What happens to the elk population in the long run? (Hint: To consider what happens in the long run, we are taking the *limit* of $P(t)$ as t goes to what value?)

In general, derivatives allow us to find rates of change. Think about your major or another topic you're interested in. What is something that could be modeled as a function? What would the derivative represent?

Additional Applications

Production Costs and Profit:

In an economics and business context, the derivative of cost is called the *marginal cost*. Suppose the cost in dollars of producing x bottles of organic chai is $c(x) = 1200 + 12x - 0.1x^2 + .0005x^3$.

(a) Find the function that represents marginal cost.

(b) What is the meaning of $c(200) = 3600$? Use units in your answer.

(c) What is the meaning of $c'(200) = 32$? Use units in your answer.

(d) If each bottle of chai is sold for \$r, write an equation for the company's profit.

(e) If the company plans to make 200 bottles of chai, how much will they need to charge per bottle to make a \$1000 profit? If they charge this amount, what is the the derivative of profit (called marginal profit) for 200 bottles?

Computer Science:

Suppose you are using a sorting algorithm to reorder a list of stored data items. The time in nanoseconds to sort n entries is given by $T(n) = 234n\log_2(n)$.

(a) Write an equation for the rate of change in sorting time with respect to number of list items.

(b) If you have a 1,000 data items, how much will the sorting time increase if you add an additional item?

(c) If another sorting algorithm takes $f(n)$ nanoseconds where $f(n) = 121n^2$, which algorithm has a greater sorting time for 1,000 data items? Which algorithm has a greater increase in sorting time if an additional item is added to those 1,000 items?

Bonus: One nanosecond is equal to one billionth of a second. Convert your solutions to units of seconds instead of nanoseconds.

Bacterial Growth:

A bacterial culture starts with 1200 bacteria in a dish of nutrient gel. The number bacteria after t minutes is $B(t) = 1200e^{0.04t}$.

1. Give a formula for the rate of change of the bacteria population in terms of t.

2. Is the bacterial population growing faster at $t = 1$ or $t = 2$? Explain your answer.

Motion of a Spring:

The function $y = A\sin\left(\left(\sqrt{\frac{k}{m}}\right)t\right)$ represents the oscillations of a mass m at the end of a spring. The constant k measures the stiffness of the spring.

(a) Write an equation for the velocity of the mass at time t.

(b) Find a value of t where the mass has velocity 0.

(c) What is the period, T of the oscillation?

(d) Find $\dfrac{dT}{dm}$. What does the sign of $\dfrac{dT}{dm}$ tell you?

Measuring pH:

To compare the acidity of different solutions, chemists use the pH (which is a single number, not the product of p and H). The pH is defined in terms of the concentration, x, of hydrogen ions in the solution as $\text{pH} = -\log_{10}(x)$.

(a) Find the rate of change of pH with respect to hydrogen ion concentration when the pH is 2.

(b) Suppose the concentration of hydrogen in a solution is equal to $h(t) = \frac{1}{10}(\sin(t)) + .5$ where t is time in hours.

  (a) Write an equation for the pH of the solution as a function of t.

  (b) Write an equation for the rate of change of the pH with respect to time. After one hour is the pH of the solution increasing or decreasing? What about after 3 hours?

SCUBA Diving:

When SCUBA diving, it's (very) important to have an accurate estimate of how much air you will consume. Air consumption depends on the volume of air in the tank - if the same amount of air takes up less volume, it will be consumed more quickly.

(a) Boyle's Law states that the volume of air in cubic feet, $V(P)$ is given by $V(P) = \frac{P_0 V_0}{P}$ where P is the pressure on the tank in *psi* and P_0, V_0 are initial measurements of pressure and volume. A SCUBA tank of 80 cubic feet is typically filled to a pressure of 3,000 *psi*. Use these intitial measurements to write an equation for volume, V in terms of pressure, P.

(b) If d is the depth a diver in meters, the pressure on the air tank, $P(d)$ is given by $P(d) \approx P_0 + 1.368d$. Use this to write an equation for volume of air in the tank as a function of depth, $V(d) = V(P(d))$.

(c) Find an equation for $V'(d)$. Include units. What does this represent?

(d) When is $V'(d)$ positive or negative? What does this mean about the effect of depth on air volume?

Health:

Suppose you conduct a study in which you measure the effect of a particular training method on the time it takes an individual to run a mile.

(a) The control group did not participate in the special training, but did exercise on their own. The study found that for a person in the control group, their mile time in minutes was approximated by $f(t)$ where t is the number of hours per week the individual exercised. $f(t) = \dfrac{40}{x+5} + 5$.

1. Find $f'(t)$. Use this to find $f'(6)$ and explain what it represents.

2. Find $\lim_{t\to\infty} f(t)$ and explain what it represents.

(b) The experimental group did participate in the special training. The study found that for a person in the experimental group, their mile time in minutes was approximated by $g(t)$ where t is the number of hours per week the individual trained. $g(t) = \dfrac{100}{\left(\sqrt{x+5}\right)^3} + 4$.

1. Find $g'(t)$. Use this to find $f'(4)$ and explain what it represents.

2. Find $\lim_{t\to\infty} g(t)$ and explain what it represents.

(c) Does the training method seem to make a difference? (If in doubt, graph!)