

Objectives:

- Review inverse functions and define inverse trigonometric functions.
- Use implicit differentiation to find derivatives of inverse trigonometric functions.

Review of Inverse Functions:

To find the inverse of a function, reverse the roles of input and output.

If $f(a) = b$ then $f^{-1}(b) = a$.

If $f(a) = f(c) = b$ for some $a \neq c$, we have a problem—what should $f^{-1}(b)$ be?

So, a function is only invertible if each input has a unique output. If a function has this nice quality, we say it is one-to-one.

Graphically, a function is invertible if it passes the horizontal line test.

If f is invertible and $f(a) = b$, then $f^{-1}(f(a)) = f^{-1}(b) = a$ and $f(f^{-1}(b)) = f(a) = b$.

Inverse Trigonometric Functions:

Let's say we want to find an inverse function for $f(x) = \sin(x)$,

which we will call $f^{-1}(x) = \arcsin(x)$.

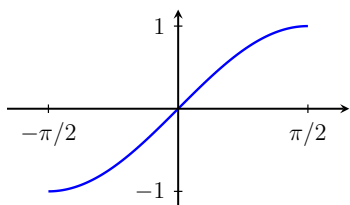
First, we need to restrict the domain of $\sin(x)$ so that we have an invertible function:

The domain we choose is $[-\pi/2, \pi/2]$.

Note: Why choose this domain?

There are other sections of the domain of $\sin(x)$ we could choose (e.g. $[\pi/2, 3\pi/2]$ or $[101\pi/2, 103\pi/2]$), but the convention is to choose $[-\pi/2, \pi/2]$ so we all agree on what we're talking about.

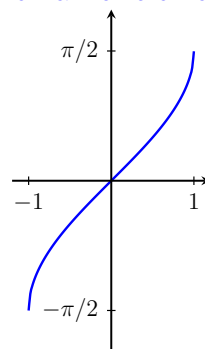
We are choosing the right half of the unit circle so each point on the half-circle has a different y -value.



$\sin(x)$, restricted

Domain: $[-\pi/2, \pi/2]$

Range: $[-1, 1]$



$\arcsin(x)$

Domain: $[-1, 1]$

Range: $[-\pi/2, \pi/2]$

Another important way of understanding this function is that $y = \arcsin(x)$ means:

1. $\sin(y) = x$

AND

2. y is between $-\pi/2$ and $\pi/2$

Now we have a new function! How can we use it?

What should the angles be in a right triangle with a hypotenuse of 3 cm and one side that is $\sqrt{3}$ cm? If θ is the angle opposite the side of length $\sqrt{3}$, $\sin(\theta) = \frac{\sqrt{3}}{3}$. Since $-1 \leq \frac{\sqrt{3}}{3} \leq 1$, we can use $\arcsin(\frac{\sqrt{3}}{3}) = \theta$ to find what θ should be. This is not a special angle, so using a calculator $\theta = \arcsin(\frac{\sqrt{3}}{3}) \approx 0.615 \approx 35.26^\circ$. Then the third angle must be about $90^\circ - 35.26^\circ \approx 54.74^\circ$

Suppose the position of a particle is given by $s(t) = \sin(t)$. When is the particle at position $\frac{1}{2}$?

$\arcsin(\frac{1}{2}) = \frac{\pi}{6}$, so at $t = \frac{\pi}{6}$, the particle is at position $\frac{1}{2}$. Since $s(t)$ is periodic, we also know $s(t) = \frac{1}{2}$ at $t = \frac{\pi}{6} + 2\pi, \frac{\pi}{6} - 2\pi, \frac{\pi}{6} + 4\pi$, etc. Further, $\sin(t) = \frac{1}{2}$ also at $\frac{5\pi}{6}$. So we have another set of solutions at $t = \frac{5\pi}{6} + 2\pi, \frac{5\pi}{6} - 2\pi, \frac{5\pi}{6} + 4\pi \dots$, etc. All of these solutions can be described by $\frac{\pi}{6} + 2\pi n$ and $\frac{5\pi}{6} + 2\pi m$, where n and m are integers.

The derivative of $\arcsin(x)$:

To find the derivative of $\arcsin(x)$, we're going to use our sneaky technique of implicit differentiation:

First: Rewrite the equation for $\arcsin(x)$ in terms of $\sin(x)$:

$$y = \arcsin(x)$$

$$\sin(y) = x$$

Second: Differentiate both sides:

$$\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$

$$\cos(y) \cdot \frac{dy}{dx} = 1$$

Third: Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

Fourth: Substitute to find a way to express $\frac{dy}{dx}$ in terms of x .

Notice that:

$$(\sin(y))^2 + (\cos(y))^2 = 1$$

$$(\cos(y))^2 = 1 - (\sin(y))^2$$

$$\cos(y) = \pm\sqrt{1 - (\sin(y))^2}$$

On the interval we care about, $y \in [-\pi/2, \pi/2]$, $\cos(y)$ is always positive, so

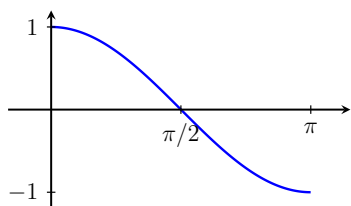
$$\cos(y) = \sqrt{1 - (\sin(y))^2}$$

So,

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - (\sin(y))^2}} = \frac{1}{\sqrt{1 - x^2}}$$

In today's activity you'll see another way to use implicit differentiation to find the derivative of a trigonometric function.

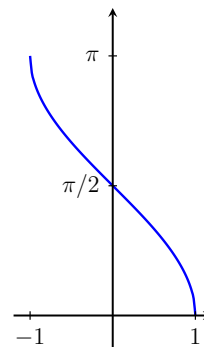
The last inverse trigonometric function we'll see is $\arccos(x)$:



$\cos(x)$, restricted

Domain: $[0, \pi]$

Range: $[-1, 1]$



$\arccos(x)$

Domain: $[-1, 1]$

Range: $[0, \pi]$

To find the derivative, we'll use implicit differentiation again.

$$y = \arccos(x)$$

$$\cos(y) = x$$

$$\frac{d}{dx}(\cos(y)) = \frac{d}{dx}(x)$$

$$-\sin(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1 - \cos^2(x)}} = \frac{-1}{\sqrt{1 - x^2}}$$

Derivatives of Inverse Trigonometric Functions:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

Examples:

- (a) Find the derivative of $f(x) = \arctan(x^3)$.

$$f'(x) = \frac{1}{1+(x^3)^2} (3x^2) = \frac{3x^2}{1+x^6}$$

- (b) $y = x^2 e^{\arcsin(x)}$. Find y' .

$$y' = x^2 (e^{\arcsin(x)})' + (2x)e^{\arcsin(x)} = x^2 (e^{\arcsin(x)}) \left(\frac{1}{1-x^2} \right) + (2x)e^{\arcsin(x)}$$