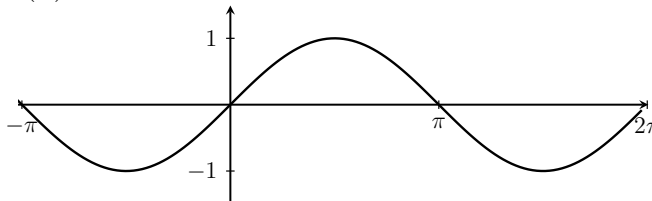


Objectives:

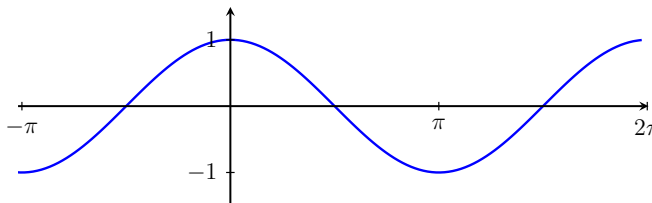
- Take derivatives of trigonometric functions.

Intuition: Let's take a look at $f(x) = \sin(x)$.

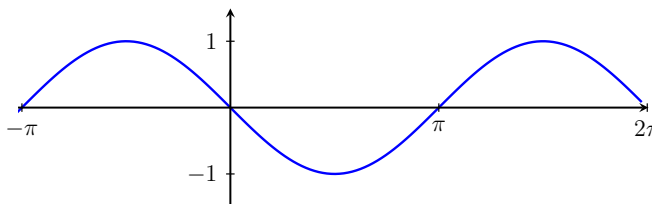
$$f(x) = \sin(x)$$



Guess: $f'(x) = \cos(x)$



Guess: $f''(x) = -\sin(x)$



Derivative of $f(x) = \sin(x)$:

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

Proof: Need $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0$ and the identity $\sin(a+b) = \sin a \cos b + \sin b \cos a$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \rightarrow 0} \frac{\sin(h)\cos(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \cos(x) \\ &= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\ &= \sin(x) \cdot 0 + \cos(x) \cdot 1 \\ &= \cos(x). \end{aligned}$$

Derivative of $f(x) = \cos(x)$:

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Proof: You will need the identity $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} + \lim_{h \rightarrow 0} \frac{\sin(x)\sin(h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \sin(x) \frac{\sin(h)}{h} \\ &= \cos(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \sin(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\ &= \cos(x) \cdot 0 - \sin(x) \cdot 1 \\ &= -\sin(x). \end{aligned}$$

More trig functions: What are the derivatives of these trig functions?

- $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ Use the quotient rule!

$$\begin{aligned} f'(x) &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x) \end{aligned}$$

- $f(x) = \sec(x) = \frac{1}{\cos(x)}$

$$\begin{aligned} f'(x) &= \frac{\cos(x) \cdot 0 - 1 \cdot (-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} \\ &= \sec(x) \tan(x) \end{aligned}$$

Derivatives of trig functions:

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

Examples: Find the derivatives of the following functions.

1. $f(x) = x^2 \sin(x)$

$$f'(x) = x^2 \cos(x) + 2x \sin(x)$$

2. $f(x) = 2^x \tan(x)$

$$f'(x) = 2^x \sec^2(x) + \ln 2 \cdot 2^x \tan(x)$$

3. $f(x) = \frac{x + \sec(x)}{\sqrt{x}}$

$$f'(x) = \frac{\sqrt{x}(1 + \sec(x)\tan(x)) - (x + \sec(x))\frac{1}{2\sqrt{x}}}{x} = \frac{1}{\sqrt{x}}(1 + \sec(x)\tan(x)) - \frac{\sqrt{x}}{2}(x + \sec(x))$$

4. $f(x) = x^2 e^x \cot x$

This is the product of three functions. Use the product rule twice.

$$f'(x) = x^2 (e^x (-\csc^2(x)) + \cot(x)e^x) + 2xe^x \cot(x) = -x^2 e^x \csc^2(x) + x^2 e^x \cot(x) + 2xe^x \cot(x).$$

Note: $(fgh)' = fgh' + fg'h + f'gh$.

5. $f(x) = \cos(\sqrt{x})$

We can't take this derivative yet since this is the composition of functions.