Objectives:

- Find the derivative of functions that are the product or quotient of other functions.

The Product Rule

If f, g are both differentiable functions:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\left(\frac{d}{dx}g(x)\right) + g(x)\left(\frac{d}{dx}f(x)\right) = f(x)g'(x) + g(x)f'(x)$$

Like all our derivative rules, we can prove the Product Rule using the limit definition of a derivative. If you'd like to prove it on your own, here's a hint:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

Examples

(a) $f(x) = x^{11}5^x$. Find $f'(x)$.

$f'(x) = x^{11}\frac{d}{dx}\left(5^x\right) + 5^x\frac{d}{dx}\left(x^{11}\right) = x^{11}\ln(5)(5^x) + 5^x(11x^{10})$

(b) $g(t) = \left(6t^5 - 3t + 1\right)e^t$. Find the derivative of $g(t)$.

$g'(t) = (6t^5-3t+1)\frac{d}{dt}(e^t)+e^t\frac{d}{dt}(6t^5-3t+1) = (6t^5-3t+1)e^t+e^t(30t^4-3) = e^t\left(6t^5 + 30t^4 - 3t - 2\right)$

Key Example: Using the Product Rule more than once. $h(t) = e^t9^tt^6$. Find $\frac{d}{dt}h(t)$

$$\begin{aligned} h'(t) &= e^t\frac{d}{dt}(9^tt^6) + (9^tt^6)\frac{d}{dt}(e^t) = e^t\left(9^t\frac{d}{dt}(t^6) + t^6\frac{d}{dt}(9^t)\right) + 9^tt^6e^t \\ &= e^t\left(9^t(6t^5) + t^6\ln(9)9^t\right) + 9^tt^6e^t = e^t9^t\left(6t^5 + \ln(9)t^6 + t^6\right) = e^t9^t\left(6t^5 + (2\ln(3)+1)t^6\right) \end{aligned}$$

The Quotient Rule

If f, g are both differentiable functions:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Note: When using the product rule, the order of $f(g')$ and $g(f')$ doesn't matter (adding either way gives the same result). This is NOT the case for subtraction, so in the quotient rule, order matters. One way to remember the ordering is "low d high minus high d low".

Later in the course we will have the tools to prove the Quotient Rule follows from the Product Rule.

For example: We can find the derivative of $h(t) = \frac{2^x}{x^3}$ using the quotient rule or the product rule.

$$h'(t) = \frac{x^3 \frac{d}{dx}(2^x) - 2^x \frac{d}{dx}(x^3)}{(x^3)^2} = \frac{x^3(\ln(2)2^x) - 2^x(3x^2)}{(x^3)^2} = \frac{x^3(\ln(2)2^x)}{x^6} - \frac{2^x(3x^2)}{x^6} = \frac{\ln(2)2^x}{x^3} - \frac{2^x(3)}{x^4}$$

Again, using the product rule instead:

$$h(t) = 2^x x^{-3}$$

$$h'(t) = 2^x \frac{d}{dx}(x^{-3}) + x^{-3}\frac{d}{dx}(2^x) = 2^x(-3x^{-4}) + x^{-3}(\ln(2)2^x) = \frac{-3(2^x)}{x^4} + \frac{\ln(2)2^x}{x^3}$$

What about the derivative of $g(x) = \frac{x^3}{2^x} = x^3 2^{-x}$? Can't use the product rule since we don't know the derivative of 2^{-x} yet. So use quotient rule:

$$g'(x) = \frac{2^x \frac{d}{dx}(x^3) - x^3 \frac{d}{dx}(2^x)}{(2^x)^2} = \frac{2^x(3x^2) - x^3(\ln(2)2^x)}{(2^x)^2} = \frac{2^x((3x^2) - x^3\ln(2))}{(2^x)^2} = \frac{(3x^2) - x^3\ln(2)}{2^x}$$

More Examples

(a) $f(x) = \frac{(3x-1)2^x}{x^3-1}$

$$f'(x) = \frac{\text{low*d(high)-high*d(low)}}{\text{low}^2} = \frac{(x^3-1)\left((3x-1)\ln(2)2^x + 2^x(3)\right) - (3x-1)(2^x)(3x^2)}{(x^3-1)^2}$$

(b) Last time we only proved the product rule for positive integer exponents.
Use the Quotient Rule to show that $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$ for any positive integer n.

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{x^n\left(\frac{d}{dx}(1)\right) - (1)\left(\frac{d}{dx}x^n\right)}{(x^n)^2} = \frac{x^n(0) - (nx^{n-1})}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$$

(c) $h(x) = f(x)g(x)$ where $f(x) = 4^x$ and the values of g, g' are given by

x	0	1	2
$g(x)$	2	5	11
$g'(x)$	3	7	19

Find $h'(0), h'(1)$, and $h(2)$.

$h'(x) = f(x)g'(x) + g(x)f'(x) = 4^x g'(x) + g(x)(\ln(4)4^x) = 4^x g'(x) + g(x)(2\ln(2)4^x)$

$h'(0) = 4^0 g'(0) + g(0)(2\ln(2)4^0) = 1(3) + 2(2\ln(2)(1)) = 3 + 4\ln(2)$

$h'(1) = 4^1 g'(1) + g(1)(2\ln(2)4^1) = 4(7) + 5(2\ln(2)(4)) = 28 + 40\ln(2)$

$h'(2) = 4^2 g'(2) + g(2)(2\ln(2)4^2) = 16(19) + 11(2\ln(2)(16)) = 304 + 352\ln(2)$