Objectives:

- Take derivatives of polynomials using the power rule, the constant multiple rule, and the sum/difference rule.

Derivative of a constant:

$$\frac{d}{dx}(c) = 0$$

Proof:

$$f(x) = c$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Derivative of $f(x) = x$:

$$\frac{d}{dx}(x) = 1$$

Proof:

$$f(x) = x$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

Derivative of $f(x) = x^2$:

$$\frac{d}{dx}(x^2) = 2x$$

Proof:

$$f(x) = x^2$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h)}{h} = \lim_{h \to 0} (2x + h) = 2x$$

Derivative of $f(x) = x^3$:

$$\frac{d}{dx}(x^3) = 3x^2$$

Proof:

$$\begin{aligned}
f(x) &= x^3 \\
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} \\
&= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\
&= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} \\
&= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h} \\
&= \lim_{h\to 0}(3x^2+3xh+h^2) \\
&= 3x^2
\end{aligned}$$

Power Rule:

For any nonzero number n, $\qquad \frac{d}{dx}(x^n) = nx^{n-1}$

Proof: (for positive integers n)

$$\begin{aligned}
f(x) &= x^n \\
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} \\
&= \lim_{h\to 0}\frac{x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n-x^n}{h} \\
&= \lim_{h\to 0}\frac{nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n}{h} \\
&= \lim_{h\to 0}\frac{h\left(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+nxh^{n-2}+h^{n-1}\right)}{h} \\
&= \lim_{h\to 0}\left(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+nxh^{n-2}+h^{n-1}\right) \\
&= nx^{n-1}
\end{aligned}$$

Constant Multiple Rule:

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}\big(f(x)\big)$$

Proof:

$$\begin{aligned}
\lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} &= \lim_{h\to 0} c\left(\frac{f(x+h) - f(x)}{h}\right) \\
&= c\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= c\frac{d}{dx}\big(f(x)\big)
\end{aligned}$$

Sum/Difference Rule:

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big) \quad \text{and} \quad \frac{d}{dx}\big(f(x) - g(x)\big) = \frac{d}{dx}\big(f(x)\big) - \frac{d}{dx}\big(g(x)\big)$$

Proof: (of the sum rule)

$$\begin{aligned}
\lim_{h\to 0} \frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h} &= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right) \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big)
\end{aligned}$$

Some practice with our new rules!

1. Find the derivatives of the following functions using our new rules:

   (a) $f(x) = 3x^5$

   $$\frac{d}{dx}(3x^5) = 3\frac{d}{dx}(x^5) = 3 \cdot 5x^4 = 15x^4$$

   (b) $f(x) = \dfrac{4}{\sqrt[3]{x}} + 3x = 4x^{-1/3} + 3x$

   $$\frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}\left(4x^{-1/3} + 3x\right) = 4\frac{d}{dx}(x^{-1/3}) + 3\frac{d}{dx}(x) = 4\left(-\frac{1}{3}x^{-4/3}\right) + 3 \cdot 1 = -\frac{4}{3}x^{-4/3} + 3$$

   (c) $f(x) = \dfrac{x^3 + \sqrt{x}}{x} = x^2 + x^{-1/2}$

   $$\frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}\left(x^2 + x^{-1/2}\right) = 2x - \frac{1}{2}x^{-3/2}$$

   (d) $f(t) = \sqrt[5]{t} + \dfrac{3}{t^2} + \dfrac{1}{4t} = t^{1/5} + 3t^{-2} + \frac{1}{4}t^{-1}$

   $$\frac{d}{dt}\big(f(t)\big) = \frac{d}{dt}\left(t^{1/5} + 3t^{-2} + \frac{1}{4}t^{-1}\right) = \frac{1}{5}x^{-4/5} - 6t^{-3} - \frac{1}{4}t^{-2}$$

   (e) $g(s) = 4s^{10} + 3s^4 + 7$

   $$\frac{d}{ds}(4s^{10} + 3s^4 + 7) = 40s^9 + 12s^3$$

   (f) $h(x) = \left(\sqrt[3]{x} + \dfrac{4}{\sqrt[3]{x}}\right)^2 = \left(x^{1/3} + 4x^{-1/3}\right)^2 = x^{2/3} + 8 + 16x^{-2/3}$

   $$\frac{d}{dx}\big(h(x)\big) = \frac{d}{dx}\left(x^{2/3} + 8 + 16x^{-2/3}\right) = \frac{2}{3}x^{-1/3} - \frac{32}{3}x^{-5/3}$$

2. Find an equation of the tangent line to $f(x) = 3x^4 + x^2 + 1$ at $x = 1$.

   $$f'(x) = 12x^3 + 2x \qquad f'(1) = 14$$

   The point $(1, 5)$ lies on the curve and the line tangent to the curve at $x = 1$ so the equation of our tangent line is

   $$y - 5 = 14(x - 1)$$