Objectives:

- Precisely define the derivative of a function at a point a and its connection to the tangent line at a.

Remember:

A ___secant line___ of a function $f(x)$ connects two points $\big(a, f(a)\big)$ and $\big(a+h, f(a+h)\big)$.

The slope of the secant line is: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(a+h)-f(a)}{(a+h)-a} = \dfrac{f(a+h)-f(a)}{h}$

This slope represents the ___average rate of change___ of the function.

As $a+h$ and a get closer together, the secant line becomes more like the ___tangent line___ at a.

The slope of a tangent line is: $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

This slope represents the ___instantaneous rate of change___ of the function at a and is called the ___derivative___ of $f(x)$ at a or ___$f'(a)$___ .

Calculating the Derivative

(See the project from recitation for more examples - solutions will be posted on the course website.)

Example: Find the derivative of $f(x) = 3x - 3x^2$ at a point a.

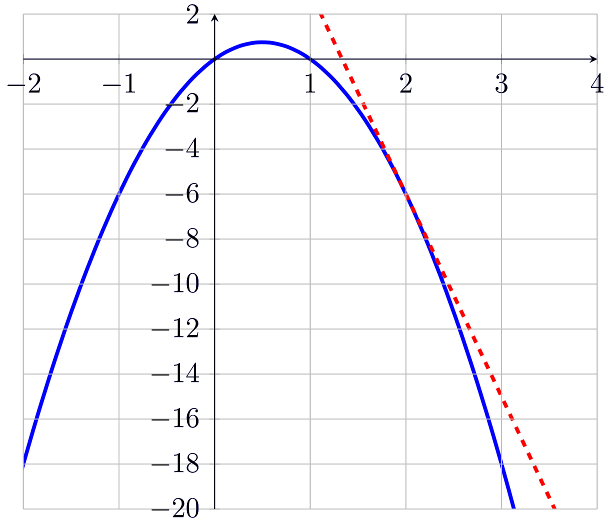
$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{3(a+h)-3(a+h)^2-[3a-3a^2]}{h} = \lim_{h\to 0} \frac{3a+3h-3(a^2+2ah+h^2)-3a+3a^2}{h} \\
&= \lim_{h\to 0} \frac{3a+3h-3a^2-6ah-3h^2-3a+3a^2}{h} = \lim_{h\to 0} \frac{3h-6ah-3h^2}{h} \\
&= \lim_{h\to 0} 3-6a-3h = 3-6a
\end{aligned}$$

Example: Find the equation of the tangent line to $f(x) = 3x - 3x^2$ at $x = 2$.

Step 1 Find the slope of the tangent line using the definition of a derivative.
From the previous example, we know $f'(a) = 3 - 6a$, so $f'(2) = 3 - 12 = -9$

Step 2 Use the slope and the point $\big(a, f(a)\big)$ to find the point-slope form of the tangent line.
Slope=-9, Point on line=$(2, f(2)) = (2, -6)$
$y - y_0 = m(x - x_0) \to y - (-6) = -9(x-2) \to y + 6 = -9(x-2)$

What does this tangent line look like? Graph $f(x) = 3x - 3x^2$ and the tangent line at $x = 2$.

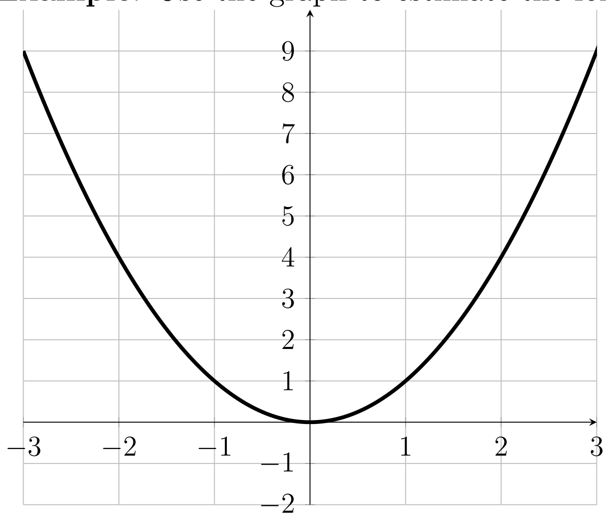


To graph the parabola: $f(x) = 3x(1-x)$ so there are roots at $x = 0$ and $x = 1$. The parabola opens down and $(2, -6)$ is a point on the curve. For the line, use $(2, -6)$ and the slope, -9.

Example: If $s(t) = 3t - 3t^2$ gives the position (ft) of an object as a function of time (min), what is the instantaneous velocity of the object at $t = 4$ minutes?

Instantaneous velocity tells us we're looking for the derivative of $s(t)$ at $t = 4$. We know $s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} = 3 - 6a$, so $s'(4) = 3 - 24 = -21$. The units are $\frac{\text{y units}}{\text{x units}} = \frac{\text{ft}}{\text{min}}$, so the instantaneous velocity at 4 minutes is -21 ft/min (i.e. moving 21 feet backwards per minute)

Example: Use the graph to estimate the following:



$f(0) \approx 0$ $f'(0) \approx 0$

$f(1) \approx 1$ $f'(1) \approx 2$

$f(2) \approx 4$ $f'(2) \approx 4$

$f(-1) \approx 1$ $f'(-1) \approx -2$