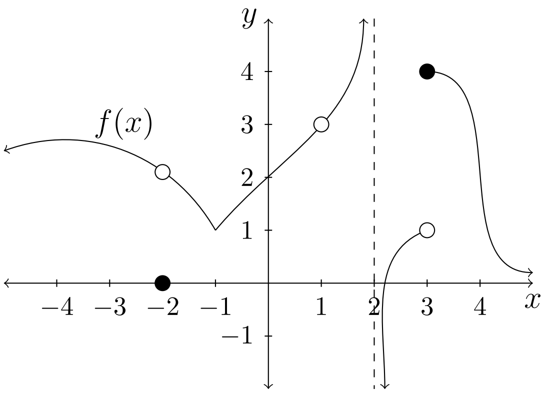
Objectives:

- Define continuity of a function (from the left, from the right, at a point, and over its domain)
- Determine if a function is continuous at a point or on its domain

Intuition: A function is continuous if you can draw its graph without lifting your pencil. This means it has no HOLES, JUMPS, or VERTICAL ASYMPTOTES.

Graphical Example:



This graph is discontinuous at

- $x =$ -2 because $\lim_{x\to -2} f(x) = 2$ but $f(x) = 0$. This is a removable discontinuity (hole). If we redefined $f(-2) = 2$, then $f(x)$ would be continuous at $x = -2$.
- $x =$ 1 because $f(1)$ is undefined. However, $\lim_{x\to 1} f(x) = 3$ so if we defined $f(1) = 3$, the function would be continuous at $x = 1$. This is another removable discontinuity (hole).
- $x =$ 2 because $f(2)$ is undefined. Since $\lim_{x\to 2^-} f(x) = \infty$ and $\lim_{x\to 2^+} f(x) = \infty$, this is an infinite discontinuity (vertical asymptote).
- $x =$ 3 because $\lim_{x\to 3^-} f(x) = 1$, $\lim_{x\to 3^+} f(x) = 4$, and $f(3) = 4$. This is a jump discontinuity. We say $f(x)$ is right continuous at $x = 4$.

Definitions:

- **CONTINUOUS:** A function $f(x)$ is continuous at a number a if

$$\lim_{x\to a} f(x) = f(a).$$

There are three requirements hidden in this definition:

1. $f(a)$ is defined;
2. $\lim_{x\to a} f(x)$ exists;
3. the above two values are equal.

If $\lim_{x\to a} f(x)$ exists, we call $x = a$ a REMOVABLE DISCONTINUITY. There are other types of discontinuities such as jumps or vertical asymptotes.

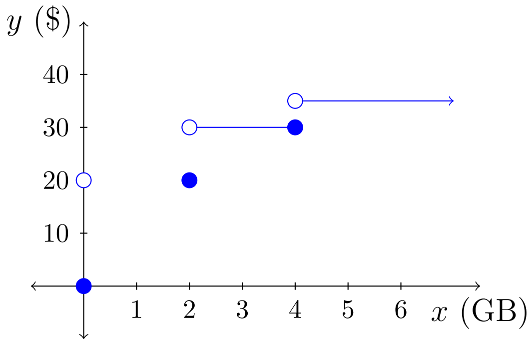
- **CONTINUOUS FROM THE RIGHT:** A function $f(x)$ is continuous from the right at a number a if

$$\lim_{x\to a^+} f(x) = f(a)$$

- **CONTINUOUS FROM THE LEFT:** A function $f(x)$ is continuous from the left at a if

$$\lim_{x\to a^-} f(x) = f(a)$$

Example: Let $p(x)$ be the price I pay for data on my cell phone plan as a function of the number of GB I purchase. If I buy 2GB or less, I pay \$20. If I buy more than 2GB but no more than 40GB, I pay \$30. If I purchase more than 40GB, I pay \$35. If I don't purchase any data plan, I don't pay anything.



$$p(x) = \begin{cases} 0 & x = 0 \\ 20 & 0 < x \leq 2 \\ 30 & 2 < x \leq 4 \\ 35 & x > 4 \end{cases}$$

The function $p(x)$ is discontinuous at $x = 0$, $x = 2$, and $x = 4$. The function is left continuous but not continuous at $x = 2$, and $x = 4$.

Question: Which functions are continuous? To answer this question, we need to think back to the direct substitution property which gives us that polynomials and rational functions satisfy

$$\lim_{x\to a} f(x) = f(a).$$

This means polynomial and rational functions are CONTINUOUS !

Conclusion: The following functions are continuous on their domains: polynomials, rational functions, root functions, trig functions, exponential functions, log functions. Also, sums, differences and products of continuous functions are continuous.

Example: Where is $f(x) = \dfrac{1}{\sqrt{5-3x}}$ continuous?

It is a quotient of a polynomial and a root function so it is continuous on its domain:
The domain of $f(x)$ is wherever $5 - 3x$ is (i) not zero and (ii) not negative.

$$\begin{aligned} 5 - 3x &> 0 \\ -3x &> -5 \\ x &< \frac{5}{3} \end{aligned}$$

So $f(x)$ is continuous on $\left(-\infty, \frac{5}{3}\right)$.