

Objectives:

- Find limits of rational functions in cases where we can't substitute
- Find limits of piecewise functions
- Define and use the Squeeze Theorem

We saw last time that if $f(x)$ is a rational function and a is in the domain of f , then $\lim_{x \rightarrow a} f(x) = f(a)$. If a is a number not in the domain of $f(x)$, trying to substitute leads to dividing by zero.

1. If trying to plug a into $f(x)$ leads to " $\frac{\text{non-zero}}{0}$ ", then there is a vertical asymptote at a . This means the one-sided limits can be ∞ or $-\infty$.

Example $\lim_{x \rightarrow 2} \frac{x+5}{x-2}$

If we try to substitute, we get " $\frac{7}{0}$ ".

Lefthand: $x < 2$, so $\lim_{x \rightarrow 2^-} \frac{x+5}{x-2} = \frac{7}{\text{tiny negative}} = \frac{+}{-} \infty = -\infty$.

Righthand: $x > 2$ so, $\lim_{x \rightarrow 2^+} \frac{x+5}{x-2} = \frac{7}{\text{tiny positive}} = \frac{+}{+} \infty = \infty$.

Since $\infty \neq -\infty$, the limit of $\frac{x+5}{x-2}$ as x goes to 2 does not exist.

Example $\lim_{x \rightarrow 0} \frac{x+1}{x^2}$

Attempting to substitute gives " $\frac{1}{0}$ ".

Lefthand: " $\frac{1}{\text{tiny positive}}$ " = $+\infty$.

Righthand: " $\frac{1}{\text{tiny positive}}$ " = $+\infty$.

So, $\lim_{x \rightarrow 0} \frac{x+1}{x^2} = \infty$.

2. If trying to plug a into $f(x)$ leads to " $\frac{0}{0}$ ", the limit is indeterminate. There are a few strategies we can try:

- (a) Factor and Cancel

Example $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \rightarrow 2} x + 2 = 4$

- (b) Combine Fractions

Example $\lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \lim_{x \rightarrow 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{x - 2} = \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{x - 2} = \lim_{x \rightarrow 2} \frac{-(x-2)}{2x} \left(\frac{1}{x-2} \right) = \lim_{x \rightarrow 2} \frac{1}{2x} = \frac{1}{4}$

(c) Multiply by the Conjugate

$$\text{Example } \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right) = \lim_{x \rightarrow 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} \\ = \frac{2}{\sqrt{4} + 2}$$

Example Let's calculate a limit that can't be approximated numerically on your calculator!

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \left(\frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \right) = \lim_{t \rightarrow 0} \frac{(t^2 + 9) - 9}{(t^2)(\sqrt{t^2 + 9} + 3)} = \lim_{t \rightarrow 0} \frac{t^2}{(t^2)\sqrt{t^2 + 9}} \\ = \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

Piecewise Functions

Not all functions are this nice! Piecewise functions require careful thinking about limit definitions:

$$f(x) = \begin{cases} -\sqrt{9+x} & -9 < x < -5 \\ 100 & x = -5 \\ x+3 & -5 < x \leq 0 \\ x^2 & 0 < x \end{cases}$$

Example 1. $\lim_{x \rightarrow -5} f(x)$: $f(-5) = 100$, but this tells us nothing about the limit.

Lefthand side: Values "just less than" -5 . These are located in $x < -5$, so

$$\lim_{x \rightarrow -5^-} f(x) = \lim_{x \rightarrow -5^-} -\sqrt{9+x} = -\sqrt{9-5} = -\sqrt{4} = -2$$

Righthand side: Values "just more than" -5 , so

$$\lim_{x \rightarrow -5^+} f(x) = \lim_{x \rightarrow -5^+} x + 3 = -5 + 3 = -2$$

Since the one-sided limits agree, $\lim_{x \rightarrow -5} f(x) = -2$.

Example 2. $\lim_{x \rightarrow 0} f(x)$:

$$\text{Lefthand side: } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x + 3 = 3$$

$$\text{Righthand side: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0$$

One-sided limits do not agree, so $\lim_{x \rightarrow 0} f(x)$ *D.N.E.*

Example 3. $\lim_{x \rightarrow -3} f(x)$: All values "near" -3 are in $-5 < x \leq 0$, so $\lim_{x \rightarrow -3} f(x) = \lim_{x \rightarrow -3} x + 3 = 0$

Don't forget $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$.

Absolute value functions require just as much caution as any other piecewise function.

Example $\lim_{x \rightarrow 0} g(x)$ where $g(x) = \frac{x}{|x|}$:

To understand what's going on, we want to write this as $g(x) = \begin{cases} -1 & x < 0 \\ x/x & x > 0 \end{cases} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

LHS: $\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} -1 = -1$

RHS: $\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} 1 = 1$

So $\lim_{x \rightarrow 0} g(x)$ D.N.E.

The Squeeze Theorem:

If $f(x) \leq g(x)$ for all x near a , then, even if $f(a) > g(a)$, we would expect that:

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

From this reasonable fact, we can deduce:

The Squeeze Theorem (a.k.a. Sandwich Theorem):

If $h(x) \leq f(x) \leq g(x)$ for all x near a (not necessarily for $x = a$),

then $\lim_{x \rightarrow a} h(x) \leq \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$

The Squeeze Theorem is useful for finding limits of weird functions by "squeezing" them with more cooperative functions:

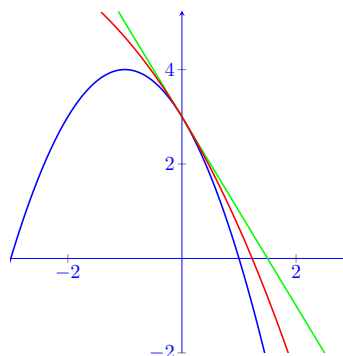
Example Let $f(x)$ be a mystery function. The only thing we know about f is $3 - 2x - x^2 \leq f(x) \leq -2x + 3$ for all $x \neq a$. Find $\lim_{x \rightarrow 0} f(x)$.

$$3 - 2x - x^2 \leq f(x) \leq -2x + 3$$

$$\lim_{x \rightarrow 0} 3 - 2x - x^2 \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} -2x + 3$$

$$3 \leq \lim_{x \rightarrow 0} f(x) \leq 3$$

So, $\lim_{x \rightarrow 0} f(x) = 3$



Example Find $\lim_{t \rightarrow 0} t^2 \sin\left(\frac{1}{t}\right)$.

We might be tempted to split up the product here, but $\lim_{t \rightarrow 0} \sin\left(\frac{1}{t}\right)$ D.N.E.

Instead, let's use the fact that $-1 \leq \sin\left(\frac{1}{t}\right) \leq 1$ to squeeze our function.

$$\begin{aligned}(-1)t^2 &\leq t^2 \sin\left(\frac{1}{t}\right) \leq (1)t^2 \\ \lim_{t \rightarrow 0}(-1)t^2 &\leq \lim_{t \rightarrow 0} t^2 \sin\left(\frac{1}{t}\right) \leq \lim_{t \rightarrow 0}(1)t^2 \\ 0 &\leq \lim_{t \rightarrow 0} t^2 \sin\left(\frac{1}{t}\right) \leq 0\end{aligned}$$

So, $\lim_{t \rightarrow 0} t^2 \sin\left(\frac{1}{t}\right) = 0$

