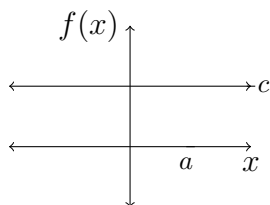


Objectives:

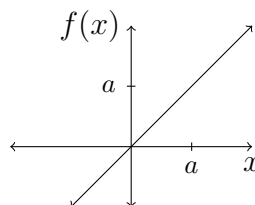
- Define some nice properties of limits (the “Limit Laws”)
- Use limit laws to compute more complicated limits

Limits we know: (These are listed in your textbook as limit laws 7 and 8.)



$$\lim_{x \rightarrow a} c = c$$

E.G. $\lim_{x \rightarrow 3} 7 = 7$



$$\lim_{x \rightarrow a} x = a$$

E.G. $\lim_{x \rightarrow 4} x = 4$

Basic Limit Laws

If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist: (This is a wildly important assumption!)

- $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- For any constant c , $\lim_{x \rightarrow a} (cf(x)) = c \left(\lim_{x \rightarrow a} f(x) \right)$
- $\lim_{x \rightarrow a} (f(x)g(x)) = \left(\lim_{x \rightarrow a} f(x) \right) \left(\lim_{x \rightarrow a} g(x) \right)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ ONLY IF $\lim_{x \rightarrow a} g(x)$ is not zero.

Some Limit Law Examples

- $\lim_{x \rightarrow 2} (x + 4) = \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 4 = 2 + 4 = 6$
- $\lim_{h \rightarrow 100} (300 - h) = \lim_{h \rightarrow 100} 300 - \lim_{h \rightarrow 100} h = 300 - 100 = 200$
- $\lim_{t \rightarrow 3} (7t) = 7 \left(\lim_{t \rightarrow 3} t \right) = 7(3) = 21$
- $\lim_{x \rightarrow 0} (x^2) = \lim_{x \rightarrow 0} (x)(x) = \left(\lim_{x \rightarrow 0} x \right) \left(\lim_{x \rightarrow 0} x \right) = (0)(0)$
- $\lim_{x \rightarrow 6} \frac{3}{x} = \frac{\lim_{x \rightarrow 6} 3}{\lim_{x \rightarrow 6} x} = \frac{3}{6} = \frac{1}{2}$ BUT we cannot use this limit law to compute $\lim_{x \rightarrow 0} \frac{3}{x}$ since $\lim_{x \rightarrow 0} x = 0$.

Building the Rest of the Limit Laws

- 6. For every positive integer n , $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x)\right)^n$ because:

$$\lim_{x \rightarrow a} (f(x))^n = \lim_{x \rightarrow a} (f(x)f(x) \dots f(x)) = \left(\lim_{x \rightarrow a} f(x)\right) \left(\lim_{x \rightarrow a} f(x)\right) \cdots \left(\lim_{x \rightarrow a} f(x)\right) = \left(\lim_{x \rightarrow a} f(x)\right)^n$$
- 7. For every positive integer n , $\lim_{x \rightarrow a} x^n = \left(\lim_{x \rightarrow a} x\right)^n = a^n$
- 8. For every positive integer n , $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$ (Note: if n is even, we need a to be positive)
- 9. For every positive integer n , $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ (if n is even, assume $\lim_{x \rightarrow a} f(x)$ is positive)
 because $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \left(\sqrt[n]{f(x)}\right)^n = \left(\lim_{x \rightarrow a} \sqrt[n]{f(x)}\right)^n$,
 so $\sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{\left(\lim_{x \rightarrow a} \sqrt[n]{f(x)}\right)^n} = \lim_{x \rightarrow a} \sqrt[n]{f(x)}$

Examples Involving Several Limit Laws

(a) Find $\lim_{x \rightarrow 1} 5x^{10} - 7x^3 - 8$.

$$\begin{aligned} \lim_{x \rightarrow 1} (5x^{10} - 7x^3 - 8) &= \lim_{x \rightarrow 1} 5x^{10} - \lim_{x \rightarrow 1} 7x^3 - \lim_{x \rightarrow 1} 8 = 5 \lim_{x \rightarrow 1} x^{10} - 7 \lim_{x \rightarrow 1} x^3 - \lim_{x \rightarrow 1} 8 \\ &= 5(\lim_{x \rightarrow 1} x)^{10} - 7 \lim_{x \rightarrow 1} x^3 - \lim_{x \rightarrow 1} 8 = 5(1)^{10} - 7(1)^3 + 1 - 8 = 5 - 7 - 8 = -10 \end{aligned}$$

(b) Find $\lim_{x \rightarrow 3} \frac{2x^3 - 1}{x^2 + 6x}$.

$$\begin{aligned} \frac{\lim_{x \rightarrow 3} (2x^3 - 1)}{\lim_{x \rightarrow 3} (x^2 + 6x)} &= \frac{\lim_{x \rightarrow 3} 2x^3 - \lim_{x \rightarrow 3} 1}{\lim_{x \rightarrow 3} x^2 + \lim_{x \rightarrow 3} 6x} = \frac{2 \lim_{x \rightarrow 3} x^3 - \lim_{x \rightarrow 3} 1}{\lim_{x \rightarrow 3} x^2 + 6 \lim_{x \rightarrow 3} x} = \frac{2 \left(\lim_{x \rightarrow 3} x\right)^3 - \lim_{x \rightarrow 3} 1}{\left(\lim_{x \rightarrow 3} x\right)^2 + 6 \lim_{x \rightarrow 3} x} = \frac{2(3)^3 - 1}{(3)^2 + 6(3)} = \\ \frac{2(27) - 1}{9 + 6(3)} &= \frac{54 - 1}{9 + 18} = \frac{53}{27} \end{aligned}$$

Conclusion: If $f(x)$ is a polynomial or rational function,
 and $f(a)$ is defined, then $\lim_{x \rightarrow a} f(x) = \underline{f(a)}$.
 We say that these functions have the direct substitution property.

Example $\lim_{t \rightarrow 2} \frac{t^3 - t + 1}{t + 1} = \frac{2^3 - 2 + 1}{2 + 1} = \frac{8 - 1}{3} = \frac{7}{3}$

Example $\lim_{t \rightarrow -1} \frac{t^3 - t + 1}{t + 1}$ cannot be found using the direct substitution property, because -1 is not in the domain of $f(t) = \frac{t^3 - t + 1}{t + 1}$.