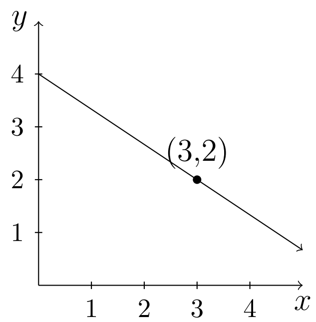
Goals:

- Define, compute, and draw secant and tangent lines.
- Interpret the slope of secant and tangent lines.

Motivating Example:



(a) Write an equation for this line and identify the slope.

$$m = \frac{2-4}{3-0} = \frac{-2}{3} \qquad y - 4 = -\frac{2}{3}(x-0) \to y = -\frac{2}{3}x + 4$$

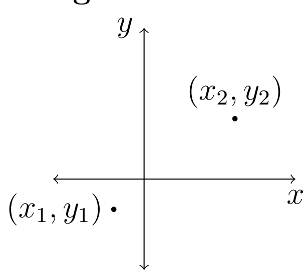
(b) If the x-values represent hours since you started hiking and the y-values represent the number of miles between you and your destination, what does the slope represent? What units should the slope be in?

The slope represents velocity in miles/hour

(c) If the x-values represent the number of books a publisher sells and the y-values represent the publisher's revenue (total amount of money received), what does the slope represent? What units should the slope be in?

The slope represents the amount of money received per book (marginal revenue) in dollars/books

Background

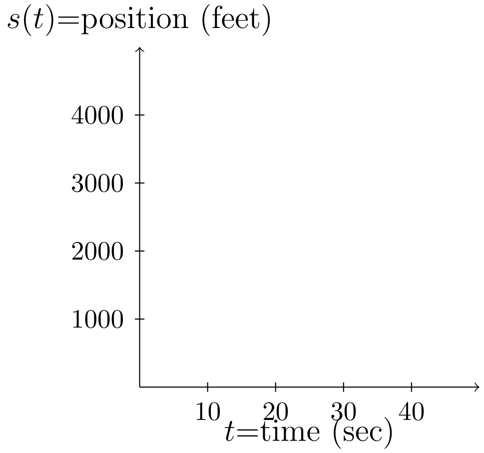


Slope $= m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Point slope form of the line: $y - y_1 = m(x - x_1)$

If $s(t)$ is a function that represents ______position______ , then the slope of the line between $\big(a, s(a)\big)$ and $\big(b, s(b)\big)$ represents ______the average velocity between times a and b______ .

Average Velocity Example 1



s	s(t)
0	200
10	500
15	1000
16	1200
20	2100
30	3200

Average velocity over $[0, 30]$:

$$\text{velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(30) - s(0)}{30 - 0}$$
$$= \frac{3200 - 200}{30} = \frac{3000}{30} = 100\frac{\text{ft}}{\text{sec}}$$

Average velocity over $[10, 20]$:

$$\frac{s(20) - s(10)}{20 - 10} = \frac{2100 - 500}{10} = \frac{1600}{10} = 160\frac{\text{ft}}{\text{sec}}$$

Average velocity over $[15, 16]$:

$$\frac{s(16) - s(15)}{16 - 15} = \frac{1200 - 1000}{1} = 200\frac{\text{ft}}{\text{sec}}$$

Which of these is the closest estimate to the velocity of the object at $t = 15$?

Average velocity over $[15, 16]$.

Write an equation for (but do not compute) an even more accurate estimate of velocity at $t = 15$:

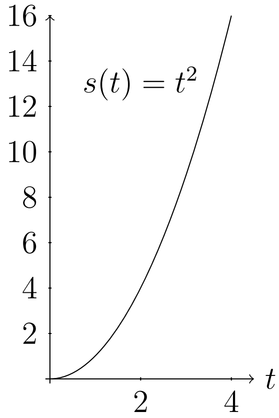
$$\frac{s(15.1) - s(14.9)}{15.1 - 14.9}$$

In general, to find the average slope of a function $f(x)$ over the interval $[a, a + h]$, compute the **difference quotient**:

$$\frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$$

[The smaller we make h, the more accurate our estimate will be. In the rest of this chapter, we're going to learn to make h as small as possible.]

Average Velocity Ex. 2



Let t be time in seconds and $s(t)$ be distance traveled in feet.

Average slope over $[2, 4]$: $\dfrac{s(4) - s(2)}{4 - 2} = \dfrac{16 - 4}{2} = \dfrac{12}{2} = 6$

A line through two points on a function is called a secant line.

The slope of the secant line between $(2, 4)$ and $(4, 16)$ is 6.

Average velocity over $[2, 3]$: $\dfrac{s(3) - s(2)}{3 - 2} = \dfrac{9 - 4}{1} = 5$

Average velocity over $[2, 2.1]$: $\dfrac{s(2.1) - s(2)}{2.1 - 2} = \dfrac{4.41 - 4}{0.1} = 4.1$

As the intervals get smaller, the secant lines get closer and closer to the tangent line at $t = 2$. The slope of the **tangent line** of $f(t)$ at $t = 2$ is exactly equal to the velocity of the object at $t = 2$.

Let's find the slope of the tangent line at $t = 2$ using the difference quotient. The average velocity between 2 and $2+h$ seconds is:

$$\frac{s(2+h) - s(2)}{(2+h) - 2} = \frac{(2+h)^2 - 3^2}{h} = \frac{4 + 4h + h^2 - 4}{h} = \frac{4h + h^2}{h} = 4 + h$$

As h gets smaller and smaller, the average velocity over $[2, 2 + h]$ gets closer and closer to 4. So, we guess that at $t = 2$, the object is traveling at 4 feet per second.

Additional Examples

(1)

Total cost of trip ($)	500	1000	1400	1600
Number of students on trip	10	20	30	40

Between 20 and 40 students, the average cost of adding one more student to the trip is: $\dfrac{C(40) - C(20)}{40 - 20} = \dfrac{1600 - 1000}{20} = \dfrac{600}{20} = 30\dfrac{\$}{\text{student}}$

(2)Compute the secant line of $f(x) = -\sqrt{x}$ between $x = 4$ and $x = 9$:

$$m = \frac{-\sqrt{9} - (-\sqrt{4})}{9 - 4} = \frac{-3 + 2}{9 - 4} = \frac{-1}{5} \qquad y - (-4) = \frac{-1}{5}(x - 2) \to y = \frac{-1}{5}x + \frac{22}{5}$$

So $\frac{-1}{5}$ is the average slope of $f(x)$ over $[4, 9]$. If this is a position over time graph, this means the average velocity is negative.