These notes were taken for a class on homological algebra at the University of Colorado, Boulder. The course was taught in the Spring 2020 semester by Jonathan Wise. The reference textbook is *An introduction to homological algebra* by Charles Weibel. The class met three days a week and one of these days was spent presenting solutions to exercises either given in class or from Weibel. These notes were, with the exception of a few arguments either missed or left for the student, taken live, and so there are bound to be errors. If you do happen to find errors or have questions, feel free to email me at juan.moreno-1@colorado.edu.

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"Homological Algebra is linear algebra in an Abelian category"

The idea is the usual one: linear algebra makes things easier. There are examples of this throughout all of mathematics. We study functions \( f : \mathbb{R} \to \mathbb{R} \) by studying their derivatives, we study manifolds by studying their tangent spaces. In each case there is a process of \textit{linearization} that makes the problem at hand more tractible. Homological algebra is the toolset that we use to linearize more abstract problems. The general setting where we apply this toolset is an abelian category.

**Definition 1.** An \textbf{abelian category} is a category \( \mathcal{C} \) such that

\begin{enumerate}[AB0) \quad \text{has finite direct sums, that is, it has finite products and coproducts, and these coincide,}
\item \( \mathcal{C} \) has kernels and cokernels,
\item images and coimages coincide.
\end{enumerate}

We note that this definition differs from that in Weibel’s text. We will elaborate on what this definition means exactly and in doing so we will reconcile our definition with that of Weibel.

Let \( \mathcal{C} \) be a category and take \( X, Y \in \text{Ob}(\mathcal{C}) \). The product of \( X \) and \( Y \), denoted in this class by \( X \times Y \), is an object of \( \mathcal{C} \) together with morphisms \( p_X : X \times Y \to X \) and \( p_Y : X \times Y \to Y \) such that the diagram

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{p_X} & X \\
\downarrow{p_Y} & & \downarrow{p_Y} \\
Y
\end{array}
\]

is universal. What this means is that if \( Z \) is an object of \( \mathcal{C} \) with morphisms \( f : Z \to Y \) and \( g : Z \to X \) then there is a unique morphism \( h : Z \to X \times Y \) such that the following diagram commutes.

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{p_Y} \\
X \times Y & \xrightarrow{p_X} & X
\end{array}
\]

As is always the case in category theory, the co-thing is obtained by reversing the arrows in the original thing. So a coproduct of objects \( X \) and \( Y \) in a category is an object, denoted \( X \sqcup Y \), with morphisms \( i_X : X \to X \sqcup Y \) and \( i_Y : Y \to X \sqcup Y \) that is universal. To be explicit, the defining property of the coproduct is that for any object \( Z \) of \( \mathcal{C} \) with morphisms \( f : X \to Z \) and \( g : Y \to Z \) there is a unique morphism \( h : X \sqcup Y \to Z \) such that the following diagram commutes.

\[
\begin{array}{ccc}
X \sqcup Y & \xrightarrow{i_Y} & Y \\
\downarrow{i_X} & & \downarrow{g} \\
X \times Y & \xrightarrow{p_X} & X \\
\downarrow{p_Y} & & \downarrow{p_Y} \\
Y & \xrightarrow{f} & Z
\end{array}
\]
In the former case, we denote the induced map \( h \) by a column vector \( \begin{pmatrix} f \\ g \end{pmatrix} : Z \to X \times Y \) and in the latter case the induced map \( h \) will be denoted by a row vector \( \begin{pmatrix} f & g \end{pmatrix} : X \sqcup Y \to Z \). More generally, suppose we have objects \( A, B, C, D \) with maps \( f_1 : A \to C, f_2 : A \to D, \) and \( f_3 : B \to C, f_4 : B \to D \). Then we have induced maps \( \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : A \to C \times D \) and \( \begin{pmatrix} f_3 \\ f_4 \end{pmatrix} : B \to C \times D \). This gives rise to a unique induced map

\[
\begin{pmatrix} f_1 & f_3 \\ f_2 & f_4 \end{pmatrix} : A \sqcup B \to C \times D.
\]

Now note that an initial object \( \emptyset \) is the product of an empty collection of objects. Similarly, a terminal object \( 1 \) is the coproduct of an empty collection of objects. It follows that in an Abelian category, the unique morphism \( \emptyset \to 1 \) is an isomorphism. We call such an object a zero object and denote it, as well as any maps into or out of it, by 0. We also obtain, for any two objects \( X \) and \( Y \) of \( \mathcal{C} \) a unique zero map \( 0 : X \to Y \) which is the composition of the unique maps \( X \to 0 \) and \( 0 \to Y \). It follows that there are induced maps \( \begin{pmatrix} id_X & 0 \\ 0 & id_Y \end{pmatrix} : X \to X \times Y \) and \( \begin{pmatrix} 0 \\ id_Y \end{pmatrix} : X \to X \times Y \). By our discussion above, we get map

\[
\begin{pmatrix} id_X & 0 \\ 0 & id_Y \end{pmatrix} : X \sqcup Y \to X \times Y
\]

and fits into the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & X \sqcup Y \\
\downarrow & & \downarrow \exists ! \\
X \times Y & \xrightarrow{\exists !} & X \times Y \\
\uparrow i_Y & & \\
Y & & \\
\end{array}
\]

Similarly, we have induced maps from the coproduct \( \begin{pmatrix} id_X & 0 \\ 0 & id_Y \end{pmatrix} : X \sqcup Y \to X \) and \( \begin{pmatrix} 0 & id_Y \end{pmatrix} : X \sqcup Y \to Y \), which by the universal property of the product gives rise to a unique map in the following commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\exists !} & X \times Y \\
\uparrow p_X & & \downarrow p_Y \\
X \sqcup Y & \xleftarrow{\exists !} & X \times Y \\
\downarrow & & \\
Y & & \\
\end{array}
\]

What condition AB0 says is that these maps are isomorphisms and inverse to one another. For condition AB1, we simply define what we mean by kernel, cokernel. While we are at it, we define what we mean by image, and coimage, then elaborate on condition AB2. These are generalizations of the corresponding notions in the category of vector spaces.

**Definition 2.** Let \( f : B \to C \) be a morphism in a category with a zero object, \( \mathcal{C} \). A kernel of \( f \), denoted \( \ker(f) \), is an object \( A \) of \( \mathcal{C} \) together with a morphism \( i : A \to B \) which is universal such that
A **cokernel** of \( f \), denoted \( \text{coker}(f) \), is an object \( D \) of \( \mathcal{C} \) with a morphism \( p : C \to D \) that is universal such that \( p \circ f = 0 \). The **image** of \( f \), denoted \( \text{im}(f) \), is defined as \( \ker(\text{coker}(f)) \), and the **coimage** of \( f \), denoted \( \text{coim}(f) \), is defined as \( \text{coker}(\ker(f)) \).

**Exercise 1.** Verify that, for any ring \( R \), the category of \( R \)-modules is an abelian category.

**Solution**

**Lecture 2: Abelian Categories (continued) - 1/15/2020**

We will continue by elaborating on condition AB2. Assuming conditions AB0 and AB1, there is a map \( \text{coim}(f) \to \text{im}(f) \) for all morphisms \( f : X \to Y \) in \( \mathcal{C} \). Let us see how to construct this map. In the last definition from last time, we defined the image and coimage of \( f \) as the kernel of the cokernel and the cokernel of the kernel, respectively. Here we think of the kernel and cokernel as not just objects, but objects with the corresponding universal map. We then have the following commutative diagram.

\[
\begin{array}{cccccc}
\ker(f) & \to & X & \xrightarrow{f} & Y & \to \text{coker}(f) \\
\downarrow & & \downarrow h & & \downarrow & \\
\text{coim}(f) & \to & \text{im}(f) & & & \\
\end{array}
\]

The dashed map, \( h \) comes from the universal property of the triangle-shaped diagram on the right and the fact that the composition \( X \xrightarrow{f} Y \to \text{coker}(f) \) is zero. We now use this map and the universal property of the coimage to construct our desired map. This amounts to showing that the composition \( \ker(f) \to X \xrightarrow{h} \text{im}(f) \) is in fact the zero map. To see this, note that for any morphism \( g : A \to B \) and object \( Z \), we have a map \( \text{Hom}(Z,\ker(g)) \to \text{Hom}(Z,A) \) given by sending \( t : Z \to \ker(g) \) to the composition \( \sigma \circ t \) where \( \sigma \) is the universal map \( \ker(g) \to X \). Now note that \( g \circ \sigma \circ t = 0 \). So the image of this map contains only morphisms \( Z \to A \) such that the post-composition with \( g \) is 0. By the universal property of the kernel, for any morphism \( Z \to A \) such that the composition with \( g \) is 0, there is a unique morphism \( Z \to \ker(g) \). Thus, the map on Hom-sets is injective. It follows that if any map \( Z \to A \) post-composes to 0 with \( g \), then \( \ker(g) \) must be the 0 element. This is indeed the case with the map \( \text{im}(f) \to Y \) so that the kernel of this map must be the zero element. It follows that since the composition

\[
\ker(f) \to X \xrightarrow{h} \text{im}(f) \to Y
\]

is the same as the composition

\[
\ker(f) \to X \xrightarrow{f} Y,
\]

which is zero, we must have that \( \ker(f) \to X \xrightarrow{h} \text{im}(f) \) is indeed zero, as desired. Condition AB2 is then simply that we require that the map \( \text{coim}(f) \to \text{im}(f) \) be an isomorphism.

**Exercise 2.** Construct the map \( \text{coim}(f) \to \text{im}(f) \) using first the universal property of \( \text{coim}(f) \).

**Solution**

That this map is an isomorphism is exactly condition AB2. Now let us look at some examples of abelian categories.

- vector spaces
• $R$-modules, for an associative algebra $R$
• diagrams in an abelian categories
• presheaves valued in an abelian category (usually)

The following proposition explains the name of an abelian category. It says that an abelian category is a category enriched in the category of abelian groups.

**Proposition 1.** If $\mathcal{A}$ is an abelian category and $X, Y$ objects in $\mathcal{A}$. Then $\text{Hom}(X, Y)$ has an abelian group structure.

**Proof.** Before we begin, we introduce some notation. Let

$$\Delta^*_x = \begin{bmatrix} id_X \\ id_X \end{bmatrix}, \quad \Delta_y = \begin{bmatrix} id_Y & id_Y \end{bmatrix}$$

Now we define the addition operation. Take morphisms $f, g : X \to Y$. This will be the composition

$$(f + g) : X \xrightarrow{\Delta^*_x} X \oplus X \to Y \oplus Y \xrightarrow{\Delta_y} Y,$$

where $X \oplus X \to Y \oplus Y$ is $\begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}$. That this operation is commutative, associative, and unital is left as an exercise.

In order to obtain inverses we need some additional results.

**Lemma 1.**

• $\ker(id_X \ 0) = \begin{bmatrix} 0 \\ id_X \end{bmatrix}$
• $\text{coker} \begin{bmatrix} id_X \\ 0 \end{bmatrix} = (0 \ id_X)$

**Proof.** That the composition

$$X \xrightarrow{\begin{bmatrix} 0 \\ id_X \end{bmatrix}} X \oplus X \xrightarrow{\begin{bmatrix} id_X & 0 \end{bmatrix}} X$$

is zero follows directly from the definitions of each of the maps since they are the canonical inclusions and projections to and from the direct sum. Then, if $f : X \to X \oplus X$ is any other map such that

$$(id_X \ 0) \circ f = 0,$$

by definition of the projection map, we have that

$$f = \begin{bmatrix} 0 \\ g \end{bmatrix},$$

for some $g : X \to X$. Evidently, then

$$f = \begin{bmatrix} 0 \\ id_X \end{bmatrix} \circ g,$$

so that the inclusion is indeed the kernel of the projection. The second bullet point follows from a symmetric argument.
We can use this result to construct an explicit negation map \(-1 : X \rightarrow X\). Let \(W = \ker(X \oplus X \xrightarrow{\Delta} X)\), and \(\sigma : W \rightarrow X\) the corresponding universal map, where
\[
\Delta = \begin{pmatrix} 1 & 1 \end{pmatrix}.
\]
We then have the following commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\sigma} & X \oplus X \\
\downarrow & & \downarrow \\
W & \xrightarrow{\sigma} & X
\end{array}
\]

**Lemma 2.** *In an abelian category, \(f : X \rightarrow Y\) is an isomorphism if and only if \(\ker(f) \cong \coker(f) \cong 0\).*

**Proof.** (TODO: the argument)

Now we define
\[
-1 = \left( (id_X \circ 0) \circ (0 \circ id_X) \right)^{-1}.
\]
One can then check that this indeed gives us a negation map so that for any \(f : X \rightarrow Y\),
\[
-1(f) + f = 0.
\]

**Exercise 3.** Show that the operation defined on the set \(\text{Hom}(X,Y)\) is commutative and associative. Show also that the zero map is the corresponding identity, and the operation is biadditive, that is, for morphisms \(f : A \rightarrow B\), \(g, h : B \rightarrow C\), and \(k : C \rightarrow D\), we have
\[
k \circ (g + h) \circ f = k \circ g \circ f + k \circ h \circ f.
\]

**Solution**

Here are some additional exercises from the textbook.

**Exercise 4.** Weibel 1.1.5 Solution

**Exercise 5.** Weibel 1.1.6 Solution

**Lecture 3: Chain Complexes - 1/17/2020**

Before we talk about chain complexes, we will introduce a category that is not quite abelian, but close and very important. Let \(A = \{\text{filtered abelian groups}\}\), that is
\[
A = \{(A, F^\cdot A) | \cdots \subset F^n A \subset F^{n+1} A \subset \cdots\}
\]
A morphism \((A, F^\cdot A) \rightarrow (B, F^\cdot B)\) is \(\varphi : A \rightarrow B\) such that \(\varphi(F^i A) \subset F^i B\). This satisfies both axioms AB0 and AB1. The product and coproduct of such a morphism \(\varphi\) ends up being just the product or coproduct of the restriction to each degree. Similarly,
\[
\ker(\varphi) = (\ker(\varphi), F^\cdot A \cap \ker(\varphi))
\]
and
\[ \text{coker}(\varphi) = (B/A, F^i(B/A)) = \text{im}(F^k(B) \to B/A). \]
this category will be extremely important when we come to talk about spectral sequences. Now for chain complexes.

**Definition 3.** Let \( \mathcal{A} \) be an abelian category. The category of **chain complexes** in \( \mathcal{A} \), denoted by \( \text{Ch}(\mathcal{A}) \), is the category whose objects are sequences
\[
\ldots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots,
\]
where the \( C_i \) are objects of \( \mathcal{A} \) and \( d_i \) morphisms in \( \mathcal{A} \). The morphisms in this category are commuting diagrams like the following
\[
\begin{array}{ccc}
\ldots & \to & A_{n+1} \to A_n \to \ldots \\
\downarrow{\varphi_{n+1}} & & \downarrow{\varphi_n} \\
\ldots & \to & B_{n+1} \to B_n \to \ldots
\end{array}
\]

**Theorem 1.** \( \text{Ch}(\mathcal{A}) \) is an abelian category if \( \mathcal{A} \) is an abelian category.

**Proof.** We use the fact that if \( I \) is any category, then the category \( \text{Hom}^+(I, \mathcal{A}) \) is abelian, where \( \text{Hom}^+ \) denotes the category of additive functors with natural transformations as morphisms. The proof follows from the fact that \( \text{Hom}^+(I, \mathcal{A}) \) is isomorphic to the category of chain complexes in the case that \( I \) is the poset category with objects integers and morphisms \( x \to y \) given by a relation \( x < y \).

**Exercise 6.** Fill in the details of the last proof.

Suppose \( A \) and \( B \) are each categories satisfying AB0. A functor \( F : A \to B \) is called additive if \( F(X \oplus Y) \cong F(X) \oplus F(Y) \).

**Exercise 7.** If \( F \) is additive then \( F(\varphi + \psi) = F(\varphi) + F(\psi) \), where \( \varphi, \psi : X \to Y \).

Suppose \( C \) is a chain complex in \( \mathcal{A} \), an abelian category. Let
\[
Z_n = \ker(d_n), \text{ and } B_n = \text{im}(d_{n+1}).
\]
Now that these are both contained in \( C_n \). We define the **n-th homology group** of \( C \) as
\[
H_n(C) = Z_n/B_n.
\]

**Remark 1.** We will switch back and forth from lower to upper indices in chain complexes using the convention \( C^n = C_{-n} \).

Note that each \( Z_n, B_n \) and \( H_n \) is functorial in \( C_n \), that is, the solid arrows below induce the dashed arrows, which in turn induce a map on homology.

\[
\begin{array}{ccc}
C_{n+1} & \to & B_n \leftarrow Z_n \leftarrow C_n \\
\downarrow{\varphi} & & \downarrow{\psi} \\
C'_{n+1} & \to & B'_n \leftarrow Z'_n \leftarrow C'_n
\end{array}
\]
Exercise 8. Construct the dashed arrows and show that there is an induced map $H_n(C) \to H_n(C')$.

Definition 4. A morphism $\varphi : C \to C'$ of chain complexes is called a **quasi-isomorphism** if $H_n(C) \to H_n(C')$ is an isomorphism for all $n$.

Example 1. The diagram to the right is a quasi-isomorphism.

Exercise 9. Weibel 1.2.5 Solution

Exercise 10. Weibel 1.2.8 Solution

Exercise 11. Weibel 1.3.1 Solution

Exercise 12. Weibel 1.3.2 Solution

Exercise 13. Weibel 1.3.5 Solution

Lecture 4: The Snake Lemma - 1/24/2020

Here is just a regular Lemma, not of the cold-blooded kind.

Lemma 3. If $\mathcal{A}$ is an abelian category and

$$X \to Y \to Z \to W$$

is an exact sequence in $\mathcal{A}$. Then for any morphism $V \to Y$,

$$X \to Y/V \to Z/V \to W$$

is also exact.

Proof. First, we give a different characterization of exactness in an abelian category. Using the sequence $X \to Y \to Z$ as a model, we have

$$\ker(Y \to Z) = \text{im}(X \to Y) \implies \text{coker}(\ker(Y \to Z)) = \text{coker}(\ker(Y \to \text{coker}(X \to Y)))$$

$$\implies \text{im}(Y \to Z) = \text{coker}(\ker(Y \to \text{coker}(X \to Y))) \text{coker}(\ker(Y \to \text{coker}(X \to Y))).$$

However, $\text{coker}(\text{coker}(X \to Y)) = \text{coker}(X \to Y)$. Therefore, another way to phrase exactness of the given sequence is to say that

$$\text{coker}(X \to Y) = \text{im}(Y \to Z), \quad \text{and} \quad \text{coker}(Y \to Z) = \text{im}(Z \to W).$$

Which is true if and only if

$$\text{coker}(X \to Y) \cong \ker(Z \to W).$$

In the above, by equality we mean that there exists a canonical isomorphism. Let us denote by $A/B$ the cokernel object of a map $B \to A$. Now

$$\text{im}(Z/V \to W) = \text{im}(Z \to W) = \text{coker}(Y \to Z) = \text{coker}(Y \to Z/V) = \text{coker}(Y/V \to Z/V).$$
The first equality above follows from the fact that $Z \to Z/V$ is an epi. The other equalities are left for the reader to ponder. This gives exactness at $Z/V$.

Now for exactness at $Y/V$, we have

$$\text{coker}(X \to Y/V) = \text{coker}(X \oplus V \to Y) = \text{coker}(V \to \text{coker}(X \to Y)) = \text{coker}(V \to \text{im}(Y \to Z))$$

$$= \text{im}(Y \to Z/V) = \text{im}(Y/V \to Z/V).$$

The reasoning behind the equalities is again left to ponder. We note that there is a rather large step going from the first to the second line above. Nevertheless, these equalities should seem plausible in the category of $R$-modules.

Let us denote by $A:B$ the kernel $\ker(B \to A)$.

**Lemma 4.** Suppose the following is an exact sequence in an abelian category $\mathcal{A}$

$$X \to Y \to Z \to W.$$  

Let $Z \to U$ be a morphism in $\mathcal{A}$. Then

$$X \to U : Y \to U : Z \to W$$

is an exact sequence.

**Proof.** $\mathcal{A}^{\text{op}}$ is an abelian category so we can apply the previous lemma.

We will record the actual Snake Lemma as a theorem

**Theorem 2.** (Snake Lemma) Suppose the diagram below commutes with exact rows and columns, then there exists the dashed arrow.

```
0 0 0 0
\downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow
K_1 \longrightarrow K_2 \longrightarrow K_3 \longrightarrow
\downarrow \downarrow \downarrow \downarrow
A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0
\downarrow \downarrow \downarrow \downarrow
0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3
\downarrow \downarrow \downarrow \downarrow
\longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_3
\downarrow \downarrow \downarrow \downarrow
0 0 0 0
```

Furthermore, the following sequence is exact

$$K_2 \longrightarrow K_3 \longrightarrow L_1 \longrightarrow L_2.$$
Proof. The main step is that the red portions of the diagram can be killed off by taking the corresponding cokernels, while the blue portions can be killed off by taking the corresponding kernels. By the preceding lemmas this process preserves exactness. We then obtain the diagram shown below this argument. Since every nonzero morphism is in between two zero morphisms, and the rows and columns are exact, we have that all the remaining nonzero morphisms must be isomorphisms. Thus

\[ \ker(L_1 \rightarrow L_2) \cong \coker(K_2 \rightarrow K_3), \]

implying the sequence

\[ K_2 \rightarrow K_3 \rightarrow L_1 \rightarrow L_2 \]

is exact.

Now let’s put this Snake Lemma to work. Suppose that I have an exact sequence of chain complexes

\[ 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0. \]

We want to obtain from this a long exact sequence on the homology of these chain complexes. First we note that \( \ker \) is a left-exact functor, and \( \coker \) is a right-exact functor. That is, if we have a commutative diagram with exact rows like below

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
B_3 : A_2 : (A_1 + K_2) \xrightarrow{\cong} B_3 : A_3 : K_2 \rightarrow 0 \\
\downarrow \\
0 \\
\downarrow \\
(L_2 : B_1 : A_1) \xrightarrow{\cong} (L_2 + B_3) : B_2 : A_1 \rightarrow 0 \\
\downarrow \\
0 \\
\downarrow \\
L_2 : L_1 \rightarrow 0 \\
\end{array}
\]

Thus the sequences

\[ 0 \rightarrow A' : A ightarrow B' : B ightarrow C' : C, \]

and

\[ A'/A \rightarrow B'/B \rightarrow C'/C \rightarrow 0 \]

are exact. We then have an exact sequence.
Proposition 2. Given a short exact sequence of chain complexes
\[ 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0, \]
there is an induced long exact sequence on homology
\[ \cdots H'_n \rightarrow H_n \rightarrow H''_n \rightarrow H'_{n-1} \rightarrow H''_{n-1} \rightarrow \cdots \]

We won’t get far today, but let’s move on to chain homotopies.
Suppose \( A, B \) are chain complexes. Define
\[ \text{Hom}_n(A, B) := \prod_{m \in \mathbb{Z}} \text{Hom}(A_m, B_{m+n}), \]
and \( d : \text{Hom}_n(A, B) \rightarrow \text{Hom}_{n-1}(A, B) \) by
\[ d(f)_m = d(f)_m + (-1)^{n-1}f_{m-1} \circ d : A_m \rightarrow B_{m+n-1}. \]

Disclaimer: The \( d \) on the right hand side of the equation above corresponds to either the differential of the complex \( A \) or \( B \), while the \( d \) on the left hand side corresponds to the new differential that we are defining.

Exercise 14. Show that \( \text{Hom}(A, B) \) forms a chain complex with differentials defined as above. Also, show that \( Z_0 \text{Hom}(A, B) = \text{Hom}(A, B) \). Solution

Exercise 15. Weibel 1.4.1 Solution

Exercise 16. Weibel 1.4.2 Solution

Exercise 17. Weibel 1.4.3 Solution

Exercise 18. Weibel 1.4.4 Solution

Exercise 19. Weibel 1.4.5 Solution

Lecture 5: Chain Homotopy - 1/27/2020

Okay, let’s talk about math. Recall from last time that if \( A, B \in \text{Ch}(\mathcal{A}) \) we can form a chain complex from the Hom-groups between these. Let \( d \) be the differential corresponding to this complex. We call a map \( f : A \rightarrow B \) nullhomotopic if \( f = d(g) \) for some \( g \in \text{Hom}_1(A, B) \). The following diagram gives a visual representation of the situation.
From the definition of the differential, we see that
\[ d(g \cdot) = dg + g_{m-1}d = f_m, \forall m. \]

The reason we call such a map nullhomotopic comes from homotopy theory. The essence though, is captured in the following result.

**Lemma 5.** If \( f : A \to B \) is nullhomotopic then \( f_* : H_n(A) \to H_n(B) \) is 0.

**Proof.** Let’s assume \( f \) is nullhomotopic, then
\[
(f_n(Z_n A)) = (dg_n + g_{n-1}d)(Z_n A) = dg_n(Z_n A) \subset B_n B. 
\]

Since \( B_n B \) gets quotiented out in homology, the result follows.

A quick corollary is that since \((f - g)_* = f_* - g_*\), if \( f \) and \( g \) are homotopic, that is to say \( f - g \) is nullhomotopic, then
\[
f_* = g_* : H_n(A) \to H_n(B).
\]

For ease of notation, we will write \( f \simeq g \) if \( f \) and \( g \) are homotopic. It is not too hard to see that this is indeed an equivalence relation and so the notation is warranted. All of this leads us to a notion of 'homotopy equivalence' of chain complexes.

**Definition 5.** We call \( A \) and \( B \) chain homotopy equivalent if \( \exists f : A \to B, g : B \to A \) such that
\[
g \circ f \simeq id_A, \quad f \circ g \simeq id_B.
\]

In this case, we call \( g \) a homotopy inverse to \( f \).

**Example 2.** If \( f : A \to B \) has a homotopy inverse then it induces an isomorphism on homology, that is \( f \) is a quasi-isomorphism. However, quasi-isomorphisms do not necessarily come induced from some chain homotopy equivalence. For a concrete example of this, consider the following diagram. The vertical arrows evidently induce an isomorphism on homology making them a quasi-isomorphism. However, it is also evident that there is no possible sequence of maps from the bottom row into the top which would compose to anything resembling of the identity.

```
\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to \cdots \\
\downarrow \quad \downarrow \quad \downarrow g \\
\cdots \to 0 \to 0 \to \mathbb{Z}/m\mathbb{Z} \to 0 \to \cdots
```

Now let’s talk a bit about cones. Suppose \( f : A \to B \) is a chain map. Then \( \text{cone}(f)_n = A_{n-1} \oplus B_n \) is a chain complex with differential given by
\[
d = \begin{pmatrix} -d & 0 \\ -f & d \end{pmatrix} : A_n \oplus B_{n+1} \to A_{n-1} \oplus B_n.
\]

The cone merges the ideas of taking a kernel and a cokernel simultaneously. To see this, recall from a previous exercise that we have an associated exact sequence of chain complexes
\[
0 \to B \to \text{cone}(f) \to A[-1] \to 0.
\]
The maps in the middle are given by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $(-1 \ 0)$, respectively. According to the Snake Lemma, this gives us a long exact sequence on homology,

$$\cdots \rightarrow H_{n+1}(A[\cdot -1]) \rightarrow H_n(B) \rightarrow H_n(\text{cone}(f)) \rightarrow H_n(A[\cdot -1]) \rightarrow H_{n-1}(B) \rightarrow \cdots.$$ 

So what does this tell us? Well note that if $H_n(\text{cone}(f)) = 0$ for all $n$, then the exact sequence implies that $f$ must be a quasi-isomorphism. It follows that $\text{cone}(f)$ is a chain complex that measures the difference between the homology of $A$ and $B$ using the map $f$ between the two. The idea is that often we don’t care about actually computing the homology of an object, we just care about comparing it to another object. The cone construction gives us an object that measures this difference in a precise way.

This is all segway into a larger topic of derived functors which we will start covering in the next lecture. For now, we provide some motivation. If $\mathcal{A}$ is an abelian category and $X \in \text{Ob}(\mathcal{A})$, then $\text{Hom}(X, \_)$ and $\text{Hom}(\_, X)$ are left-exact functors. However, they are not always right-exact depending on the object $X$. That is, if

$$0 \rightarrow Y \rightarrow W \rightarrow Z \rightarrow 0$$ 

is exact, then

$$0 \rightarrow \text{Hom}(X,Y) \rightarrow \text{Hom}(X,W) \rightarrow \text{Hom}(X,Z) \rightarrow 0$$

is not necessarily exact at $\text{Hom}(X,Z)$. We would like to measure the extent to which they are not right-exact for a given $X$. Specifically, taking a hint from the preceding discussion about mapping cones, we might hope to find an object $E(X)$, which would depend on $X$, for which the sequence

$$0 \rightarrow \text{Hom}(X,Y) \rightarrow \text{Hom}(X,W) \rightarrow \text{Hom}(X,Z) \rightarrow E(X)$$

is exact. Well this is actually not very difficult, we can simply take the cokernel of the last map and we would be done. However, this does not tell us much. Instead, we might hope to find a sequence of objects $E_n(X)$, which would also depend on $X$, for which the sequence

$$0 \rightarrow \text{Hom}(X,Y) \rightarrow \text{Hom}(X,W) \rightarrow \text{Hom}(X,Z) \rightarrow E_1(X) \rightarrow E_2(X) \rightarrow \cdots$$

forms a chain complex. Then the homology of such a chain complex would provide some new information. The hope is that by choosing the spaces $E_n(X)$ wisely, we will get information about the failure of the functors $\text{Hom}(X,\_)$ to be right-exact. It turns out that good choices $E_n$ exist and these give our first examples of derived functors.

Before moving on to constructing such functors, let’s talk about objects for which right-exactness of these Hom-functors does hold.

**Definition 6.** An object $P \in \mathcal{A}$ is called **projective** if the functor $\text{Hom}(P, \_)$ is exact.

**Example 3.** $\mathbb{Z}/m\mathbb{Z}$ is not projective in $\text{Ab}$. 

**Example 4.** All free modules in $\mathbb{R} - \text{mod}$ are projective.

**Proposition 3.** The following are equivalent

- $P$ is projective.
- every exact sequence $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$ can be split.
- Given a diagram as the one below, there exists a dahed arrow making the diagram commute.
Additionally, if \( \mathcal{A} \) is the category of \( R \)-modules. Then we also have that the above are equivalent to

- \( P \) is a direct summand of a free module.

**Exercise 20.** Weibel 1.5.3 Solution.

**Exercise 21.** Weibel 1.5.4 Solution.

**Exercise 22.** Weibel 1.5.8 Solution.

**Lecture 6: Projective Objects - 1/31/2020**

We will pick up where we left off yesterday by proving the following portion of the last proposition.

**Proposition 4.** Free modules are projective.

**Proof.** Suppose \( F \) is free on a set \( X \). Then

\[
\text{Hom}(F, M) = \text{Hom}_{\text{set}}(X, M).
\]

If \( B \to C \) is injective, then

\[
\begin{array}{ccc}
\text{Hom}(F, B) & \longrightarrow & \text{Hom}(F, C) \\
\downarrow & & \downarrow \\
\text{Hom}(X, B) & \longrightarrow & \text{Hom}(X, C)
\end{array}
\]

This morphism is injective by the Axiom of Choice. This is an important thing to notice: projectivity is related to the axiom of choice. This is in a sense the key property of projective objects as we will see later.

**Proposition 5.** A direct summand of a projective object is projective.

**Proof.** Suppose \( F = P \oplus Q \) with \( F \) projective. We know that we have a a diagram like the following by projectivity of \( F \).

\[
\begin{array}{ccc}
F & \downarrow \\
B & \to & C \to 0
\end{array}
\]

However, we can now form the composition with the canonical inclusion

\[
P \to F \to B,
\]

and we have our desired lift.

**Proposition 6.** In \( R \text{-mod} \), \( P \) is projective if and only if \( P \) is a direct summand of a free module.
Proof: If \( P \) is a direct summand of a free module then this follows from the previous proposition. For the reverse implication, the key is that for any module, in particular for our \( P \), there is some free module \( F \) such that \( P \) is a quotient of \( F \), that is, there is a surjection \( F \to P \). 

We will capture this important property of the \( R \)-module category used in the last proof into a definition.

**Definition 7.** \( \mathcal{A} \) has **enough projectives** if every \( X \in \mathcal{A} \) is a quotient of a projective.

At this point, Weibel moves on to injective objects. However, we will skip ahead a bit to build up some more motivation. If \( \mathcal{A} \) has enough projectives, and \( M \in \mathcal{A} \), build a resolution of \( M \) by projectives as follows:

- choose a projective \( P_0 \) and a surjection \( P_0 \to M \).
- choose a projective \( P_1 \) and a surjection \( P_1 \to \ker(P_0 \to M) \)
- proceed in this way to choose \( P_n \).

We then obtain a long exact sequence

\[
\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.
\]

What we have accomplished here is to *unwind* some, possibly complicated, object \( M \) into a sequence of projective objects. We note that this is in fact equivalent to having a chain complex \( P \cdot \) of projective objects such that \( P_n = 0 \) for \( n < 0 \) and a chain map \( P \to M \cdot \) where \( M \cdot \) is the chain complex with \( M_0 = M \) and \( M_n = 0 \) for all \( n \neq 0 \). Lets look at an example.

**Example 5.** Consider the exact sequence below which is an example of a projective resolution.

\[
0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0
\]

Apply the functor \( \text{Hom}(\_ , \mathbb{Z}/m\mathbb{Z}) \) to the sequence above, recalling that this is a left-exact functor, we get an exact sequence

\[
0 \to \text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \to \text{Hom}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{m} \text{Hom}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}).
\]

This is equivalent to the sequence

\[
0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{1} \mathbb{Z}/m\mathbb{Z} \xrightarrow{0} \mathbb{Z}/m\mathbb{Z}.
\]

If we consider the problem of extending this to the right as an exact sequence, we can simply add the cokernel of the last map on the right, namely, \( \mathbb{Z}/m\mathbb{Z} \). Note, however, that if we consider the chain complex of the projective resolution

\[
0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0
\]

and apply the \( \text{Hom} \)-functor to get

\[
0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{0} \mathbb{Z}/m\mathbb{Z} \to 0
\]

and then consider the homology of this complex, we get

\[
H_n = \begin{cases} 
\mathbb{Z}/m\mathbb{Z}, & \text{if } n = 1, 0 \\
0, & \text{else}
\end{cases}
\]

That the cokernel shows up in this homology is no coincidence.
Now let us talk about injective objects, which are precisely dual to projective objects.

**Definition 8.** \( I \in \mathcal{A} \) is **injective** if \( I^{\text{op}} \in \mathcal{A}^{\text{op}} \) is projective.

In this case, we have an equivalent definition given by the following diagram.

\[
\begin{array}{ccc}
0 & \to & A \\
& & \downarrow \gamma \\
& I & \to B
\end{array}
\]

We have a similar result for injectives, but this is harder to prove so we will record this as a theorem:

**Theorem 3.** \( R \text{- mod} \) has enough injectives, that is, every \( R \)-module can be embedded in an injective.

**Exercise 23.** If \( \mathcal{A} = \text{Ab} \), show that \( M \in \mathcal{A} \) is projective if and only if it is free and injective if and only if its divisible. Here divisibility means multiplication by an element of \( R \) is surjective. Weibel uses this to prove that there are enough injectives in \( \text{Ab} \).

**Solution.**

**Remark 2.** The same actually holds for \( R \)-modules over a PID \( R \).

**Definition 9.** Suppose we have a functor \( F : \mathcal{A} \to \mathcal{B} \) of abelian categories. We call \( F \) **additive** if

\[
F(X \oplus Y) \to F(X) \oplus F(Y)
\]

is an isomorphism. This implies that the induced map on Hom-sets is a group homomorphism.

A functor \( G : \mathcal{A} \to \mathcal{B} \) is a **left-adjoint** to \( F \) if there exists an isomorphism

\[
\text{Hom}(GX,Y) \to \text{Hom}(X,FY),
\]

that is natural in \( X \) and \( Y \).

If you have not come across the notion of adjoint functors before, the canonical example of such pairs are the free and forgetful functors.

**Example 6.** Let

\[
G : \text{Sets} \to R \text{- mod}
\]

be the functor taking \( X \) to the free module generated by \( X \), and let

\[
F : R \text{- mod} \to \text{Sets}
\]

be the functor taking an \( R \)-module \( M \) to its underlying set \( M \).

**Example 7.** Define a functor

\[
F : R \text{- mod} \to \text{Ab}
\]

by \( F(M) = M \), the underlying abelian group. Then

\[
\text{Hom}(A,M) = \text{Hom}(R \otimes A, M).
\]

So taking a tensor product with \( R \), \( A \to R \otimes A \), is a left adjoint to \( F \). But \( F \) also has a right adjoint given by

\[
G(A) = \text{Hom}_{\text{Ab}}(R, A).
\]

**Exercise 24.** Weibel 2.3.2 Solution.
Recall that an injective object is one such that for a given solid arrow diagram below there exists a dashed arrow.

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \\
 & I & \end{array}
\]

Now we give a sufficient condition for a functor to send injective objects to injective objects.

**Proposition 7.** Let \( F : \mathcal{A} \rightarrow \mathcal{B} \) be an additive functor between abelian categories and \( F \) has a left-exact left adjoint, \( G \). Then \( F \) preserves injectives.

**Proof.** Assume that \( I \in \mathcal{A} \) is injective. We want to show that \( \text{Hom}_\mathcal{B}(\_ , F(I)) \) is exact. Since \( G \) is the left adjoint to \( F \), this is the same as \( \text{Hom}_\mathcal{A}(G(\_), I) \). However, this is now a composition of the exact functors \( \text{Hom}_\mathcal{A}(\_ , I) \) and \( G \), and so it must be exact. \( \square \)

Let \( \mathcal{A} = \text{Ab} \) and \( \mathcal{B} = \text{R-mod} \). Let \( F(A) = \text{Hom}_\text{Ab}(R, A) \) and \( G(M) = M \) be the underlying abelian group. Note that \( G \) is exact. We claim that \( F \) is right-adjoint to \( G \) so \( F \) must preserve injectives. The \( R \)-module structure on \( \text{Hom}_\text{Ab}(R, A) \) is given by

\[
(\lambda \cdot \varphi)(x) = \varphi(x\lambda), \quad \forall x, \lambda \in R.
\]

The adjunction comes from the following bijections

\[
\text{Hom}_\text{R-mod}(N, \text{Hom}_\text{Ab}(R, A)) \overset{u}{\rightarrow} \text{Hom}_\text{Ab}(N, A)
\]

given by

\[
u(\varphi)(x) = \varphi(x)(1), \]

and

\[
v : \text{Hom}_\text{Ab}(N, A) \overset{v}{\rightarrow} \text{Hom}_\text{R-mod}(N, \text{Hom}_\text{Ab}(R, A)),
\]

given by

\[
v(\psi)(x)(y) = \psi(yx).
\]

We leave it to the reader to verify that these are in fact inverses. You should do this exercise because one should enjoy those moments when exact formulas are given!

We now have the tools to return to the proof of Theorem 3.

**Proof.** (Theorem 3) Suppose \( M \) is an \( R \)-module. Then, by a Exercise 22, \( \exists i : M \rightarrow I \) with \( I \) a divisible abelian group, and \( M \) denoting the underlying abelian group of \( M \). Then \( i \in \text{Hom}_\text{Ab}(M, I) \) so we can use the bijection \( v \) defined above to get a map

\[
v(i) : M \rightarrow \text{Hom}_\text{Ab}(R, I).
\]

The target of this morphism is injective, so it remains to show that this map is injective. This follows simply from noting that the composition

\[
M \rightarrow \text{Hom}_\text{Ab}(R, I) \overset{\text{ev}_I}{\rightarrow} I,
\]

is nothing but \( i \). This is injective and so \( v(i) \) must be injective. \( \square \)
Now we address our original motivation for these projective and injective objects and define derived functors.

**Definition 10.** Suppose $F : \mathcal{A} \to \mathcal{B}$ is a right-exact functor. Define $L_nF(X)$ for $X \in \mathcal{A}$ by the following process:

1. choose a projective resolution $P \xrightarrow{\cong} X$, where the $\cong$ symbol means quasi-isomorphism.
2. Set $L_nF(X) = H_nF(P)$.

We will show that $L_nF(X)$ is well-defined up to a canonical isomorphism and so the choice of projective resolution will be immaterial.

**Proposition 8.** If $P$ is a projective resolution of $M$ and $Q$ is a resolution (not necessarily projective) of $N$, then there is an isomorphism $H_0(\text{Hom}(P, Q)) \xrightarrow{\cong} \text{Hom}(M, N)$.

**Proof:** First, recall that $Z_0\text{Hom}(P, Q) = \text{Hom}(P, Q)$ is just the set of chain maps. Let $f \in \text{Hom}(M, N)$. Then we have the following diagram of solid arrows where the rows are the given resolutions in the proposition statement. Then by projectivity of $P_0$, we have the existence of the dashed morphism which we denote by $\varphi_0$.

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

$$\downarrow \varphi_0 \downarrow f$$

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

Now repeat this process replacing $P_0$ and $Q_0$ with their cycles to obtain a map $\varphi_1 : P_1 \to Q_1$.

$$\cdots \longrightarrow P_1 \longrightarrow Z(P_0) \longrightarrow 0$$

$$\downarrow \varphi_0 \downarrow \varphi_0$$

$$\cdots \longrightarrow Q_1 \longrightarrow Z(Q_0) \longrightarrow 0$$

Continuing in this way gives us a map of chain complexes $\varphi : P \to Q$. We now show that this induced chain map is in fact unique up to chain homotopy. Suppose $f = 0$. We must then show that $\varphi$ is nullhomotopic. In this case, we have the following diagram, similar to the previous one. The dashed arrow here exists by the universal property of the kernel $ZQ_0$.

$$P_0 \longrightarrow M \longrightarrow 0$$

$$\downarrow \varphi_0 \downarrow 0 \downarrow 0$$

$$ZQ_0 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

We then have the following diagram in which the projective property of $P_0$ is again used to obtain the dashed map $\sigma_0$.

$$P_0 \longrightarrow ZQ_0 \longrightarrow 0$$
Commutativity of the diagram then tells us that
\[ \varphi_0 = d \circ \sigma_0 = d \circ \sigma_0 + \sigma_{-1} \circ d, \]
with \( \sigma_{-1} = 0 \).

We now proceed similarly and consider the diagram below, where
\[ \alpha = \varphi_1 - \sigma_0 \circ d, \]
and
\[ \beta = d \circ \varphi_1 + d \circ \sigma_0 \circ d = d \circ \varphi_1 + \varphi_0 \circ d. \]

The unique dashed map again exists by the universal property of the kernel. Repeating the same process as above then gives us a diagram like the following where the dashed map exists by projectivity of \( P_1 \). Then by commutativity of the diagram, we have that \( d \circ \sigma_1 = \alpha = \varphi_1 - \sigma_0 \circ d \), implying that \( \varphi_1 = d \circ \sigma_1 + \sigma_0 \circ d. \)

Continuing by induction gives us maps \( \sigma_n \) such that \( \varphi_n = d \circ \sigma_n + \sigma_{n-1} \circ d \), implying chain map \( \varphi \) is nullhomotopic.

We now use this to show that \( L_nF(M) \) is well-defined. Suppose \( P \) and \( Q \) are projective resolutions of \( M \). We then obtain a chain map \( \varphi : P \rightarrow Q \) which is the image of \( 1_M \in \text{Hom}(M, M) \) under the isomorphism constructed in Proposition 8. This chain map then induces a homomorphism \( \varphi_* : H_nF(P) \rightarrow H_nF(Q) \). Furthermore, if \( \psi : P \rightarrow Q \) is another lift of \( 1_M \), then the preceding proposition implies that \( \varphi \simeq \psi \) so that \( \varphi_* = \psi_* \). This shows that this induced homomorphism is canonical. To see that this is in fact an isomorphism, note that since both \( Q \) and \( P \) are projective resolutions, we can also induce a chain map in the opposite direction, \( \eta : Q \rightarrow P \). We then have the following commutative diagram.

It follows that \( \eta \circ \varphi : P \rightarrow P \) lifts the identity \( 1_M \) so by Proposition 8 we have that this must in fact be chain homotopic to the identity \( 1_P \), implying that \( (\eta \circ \varphi)_* = \eta_* \circ \varphi_* = 1 \). Similarly, \( \varphi \circ \eta \simeq 1_Q \) so that \( \varphi_* \circ \eta_* = 1 \). Thus, \( H_nF(P) \) and \( H_nF(Q) \) are canonically isomorphic, and so the left derived functors \( L_nF(M) \) are well-defined up to this canonical isomorphism.
Recall that projective resolutions are unique up to homotopy. That is, if $P$ and $Q$ are projective resolutions of $M$. Then there is a chain map $\varphi : P \to Q$, lifting the identity $1_M$, which is unique up to chain-homotopy. We ended last lecture with showing that this chain map induces an isomorphism on homology. So we have that for any right exact functor $F$, $H_n F(P) \cong H_n F(Q)$. We also saw that $\varphi$ is unique up to chain homotopy so this isomorphism is in fact canonical. This allows us to think of $H_n F(P)$ as being independent of the choice of projective resolution $P$, and we call this object the left derived functor $L_n F(M)$.

Notice that right exactness was not used at all in the above. What right-exactness gives us is that $L_0 F(M) = \text{coker}(F(P_1) \to F(P_0)) = H_0 F(P) = F(\text{coker}(P_1 \to P_0)) = F(M)$.

So $L_0 F = F$. Now let’s look at some examples.

**Example 8.** Fix $N \in \mathbf{R} - \text{mod}$. For $M \in \mathbf{R} - \text{mod}$, define $F(M) = N \otimes_{\mathbb{R}} M$. This is right-exact (you can look in Weibel for a verification of this). Assume $R = \mathbb{Z}$, $M = \mathbb{Z}/n\mathbb{Z}$. Then we have a projective resolution

\[ 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0 \]

Now

\[ L_n F(\mathbb{Z}/m\mathbb{Z}) = H_n F([\mathbb{Z} \xrightarrow{m} \mathbb{Z}]) = H_n ([N \xrightarrow{m} N]) = \begin{cases} 0, & n \neq 0, 1 \\ \mathbb{N}/m\mathbb{N}, & n = 0 \\ \ker(m), & n = 1 \end{cases} \]

So we have that $L_n F(\mathbb{Z}/n\mathbb{Z}) = \text{Tor}_n (N, \mathbb{Z})$ computes torsion.

**Example 9.** Let $F(M) = \text{Hom}(M, N)$. Note that this is contravariant, left-exact functor. So we can think of this as a sort of ‘right derived functor’ $\mathbf{R} - \text{mod} \to \mathbf{Ab}^{\text{op}}$.

The reason these derived functors are useful is because they are computable. The reason they are computable is the same that ordinary homology is, namely, they have long exact sequences associated to them.

**Theorem 4.** Suppose $F : \mathcal{A} \to \mathcal{B}$ is a right-exact, additive functor between abelian categories and $\mathcal{A}$ has enough projectives. If

\[ 0 \to M' \to M \to M'' \to 0 \]

is exact in $\mathcal{A}$. Then there is a long exact sequence

\[ \cdots \to L_n F(M) \to L_n F(M'') \xrightarrow{\delta} L_{n-1} F(M') \to L_{n-1} F(M) \to \cdots \]

where $\delta$ is characterized uniquely.

We will make the final part of that statement more precise later. The following result is often called the Horseshoe Lemma.

**Lemma 6.** Suppose

\[ 0 \to M' \to M \to M'' \to 0 \]

is exact and $P', P''$ are projective resolutions of $M'$ and $M''$. Then there is a projective resolution $P$ of $M$ with $P_n = P'_n \oplus P''_n$ and a commutative diagram with exact rows.
You can refer to Weibel for a nice proof of this result and also a nice diagram which explains the name of this result. We will now use this to construct the long exact sequence of the theorem above. Choose $0 \to P' \to P \to P'' \to 0$ as in the horeshoe lemma. Then the sequence

$$0 \to F(P') \to F(P) \to F(P'') \to 0$$

is exact since $F(P_n) = F(P'_n \oplus P''_n) = F(P'_n) \oplus F(P''_n)$ and $F$ is a right-exact functor. We now get a long exact sequence on homology

$$\cdots \to H_n F(P) \to H_n F(P'') \to H_n F(P') \to H_{n-1} F(P) \to \cdots .$$

This is not quite a proof of Theorem 4 above since there are many details to check. We refer the reader to Weibel's text for those details.

Now, as usual in mathematics, we will extract the nice properties of these functors into a definition of nice objects.

**Definition 11.** A $\delta$-functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is a sequence of functors $F_n : \mathcal{A} \to \mathcal{B}$ such that if

$$0 \to M' \to M \to M'' \to 0$$

is exact in $\mathcal{A}$ then there are natural transformations

$$F_n(M'') \xrightarrow{\delta} F_{n-1}(M')$$

fitting into a long exact sequence like the one above.

Our hard work above shows that the left derived functors of a right-exact additive functor form a $\delta$-functor.

**Definition 12.** A morphism of $\delta$-functors $F$ and $G$ is a sequence of natural transformations $\varphi_n : F_n \to G_n$ such that the diagram below commutes

$$\begin{array}{ccc}
F_n(M'') & \xrightarrow{\delta} & F_{n-1}(M') \\
\downarrow{\varphi_n} & & \downarrow{\varphi_n} \\
G_n(M'') & \xrightarrow{\delta} & G_{n-1}(M')
\end{array}$$

**Definition 13.** A $\delta$-functor $F$ is called universal if

$$\text{Hom}_{\delta}(G,F) = \text{Hom}(G_0,F_0),$$

that is every natural transformation $G_0 \to F_0$ extends uniquely to a morphism of $\delta$-functors.

These universal $\delta$-functors come in handy for showing that the sequence above is independent of the many choices we made along the way.

**Exercise 25.** Weibel 2.4.3 Solution

**Exercise 26.** Weibel 2.5.1 Solution

**Exercise 27.** Weibel 2.5.2 Solution

**Exercise 28.** Weibel 3.1.1 Solution

**Exercise 29.** Weibel 3.1.2 Solution

**Exercise 30.** Weibel 3.1.3 Solution
Last time we started talking about $\delta$-functors as an alternate way of viewing derived functors. We will see this to its end and by the end we will have characterized derived functors in a universal property. First note that we can view $\delta$-functors $F : \mathcal{A} \to \mathcal{B}$ as a functor from the category of short exact sequences in $\mathcal{A}$ to the category of long exact sequences in $\mathcal{B}$.

**Definition 14.** A $\delta$-functor $F : \mathcal{A} \to \mathcal{B}$ is called **coffeaceable** if for all $M \in \mathcal{A}$ there exists $P \in \mathcal{A}$ and an epi $P \to M$ such that the induced map $F_n(P) \to F_n(M)$ is the zero map for all $n > 0$.

**Definition 15.** A $\delta$-functor $F$ is called **universal** if

$$\text{Hom}(G \cdot F) \to \text{Hom}(G_0, F_0)$$

is a bijection for all $\delta$-functors $G$.

Often times it is easy to show that universal objects exist. However, a more difficult thing is to describe what they look like. The following result, which is actually Exercise 2.4.5 in Weibel, tells us what universal $\delta$-functors look like.

**Theorem 5.** If $F$ is coffeceable then it is universal.

**Proof:** Assume $G_0 \to F_0$ is a natural transformation, and $F$ is coffeceable. We want to construct a natural transformation $G_n \to F_n$ by induction on $n$. So we will assume that $G_m \to F_m$ is constructed for $m < n$ and that it is a morphism of $\delta$-functors where defined. The base case $n = 0$ is given. We now have only one option: to 'coffeace the values of $M$'. Chose an epi $P \to M$ such that $F_n(P) \to F_n(M)$ is the zero map for all $n > 0$. Let

$$N = \ker(P \to M),$$

so that we have a short exact sequence

$$0 \to N \to P \to M \to 0.$$

We then have the following diagram connecting the long exact sequences associated to $F$ and $G$ with the solid vertical maps coming from our inductive hypothesis.

$$
\begin{array}{cccccccc}
G_n(P) & \longrightarrow & G_n(M) & \delta & G_{n-1}(N) & \longrightarrow & G_{n-1}(P) \\
& \downarrow^{u_n} & \downarrow^{u_{n-1}} & & \downarrow^{u_{n-1}} & & \\
F_n(P) & \longrightarrow & F_n(M) & \delta & F_{n-1}(N) & \longrightarrow & F_{n-1}(P)
\end{array}
$$

The dashed arrow comes from the fact that the composition

$$G_n(M) \delta \xrightarrow{u_{n-1}} G_{n-1}(N) \longrightarrow F_{n-1}(N) \to F_{n-1}(P)$$

is zero and, by exactness of the bottom row, $\delta : F_n(M) \to F_{n-1}(N)$ is the kernel of $F_{n-1}(N) \to F_{n-1}(P)$. Now that we have the map $u_n : G_n(M) \to F_n(M)$, we must show three things: (i) it is independent of $P$, (ii) it is natural, (iii) it commutes with $\delta$. First we note that this construction is functorial in $P$, that is given a diagram like the following.

$$
\begin{array}{cccccccc}
0 & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & M' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$
we get an induced commutative square

\[
\begin{array}{ccc}
G_n(M') & \longrightarrow & G_n(M) \\
\downarrow u_n & & \downarrow u_n \\
F_n(M') & \longrightarrow & F_n(M)
\end{array}
\]

This implies that the construction is independent of \(P\), because then we can choose \(Q\) to coefface the fiber product \(P \times_M P'\). (Insert diagram and show that naturality follows from this as well). For (iii), choose coeffacing \(P \to M\) and assume we have a short exact sequence

\[
0 \to M' \to M \to M'' \to 0
\]

Then we have

\[
\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}
\]

Now the idea is to relate \(\delta\) for the top row and bottom row to get the following diagram.

\[
\begin{array}{ccccccc}
G_n(M'') & \longrightarrow & \delta & \longrightarrow & G_n(N) \\
\downarrow & & & & \downarrow \\
G_n(M) & \longrightarrow & \delta & \longrightarrow & G_n(M') \\
\downarrow u_n & & \downarrow u_{n-1} & & \downarrow \\
F_n(M'') & \longrightarrow & \delta & \longrightarrow & F_{n-1}(M') \\
\downarrow & & & & \downarrow \\
F_n(M'') & \longrightarrow & \delta & \longrightarrow & F_{n-1}(N)
\end{array}
\]

(todo: finish/clear up this proof).

\[
<++>
\]

**Example 10.** Let \(N \in \text{mod} - R\) and \(M \in R - \text{mod}\). Recall that

\[
\text{Tor}(N,_) = L_n(N \otimes _-).
\]

If \(M\) is projective, then \(M\) is a projective resolution to itself,

\[
L_n(N \otimes M) = \begin{cases} 
N \otimes M, & n = 0 \\
0, & n \neq 0
\end{cases}
\]

This implies \(\text{Tor}(M,_)\) is coeffaceable: for any \(M\) choose an epi \(P \to M\) with \(P\) projective, then \(\text{Tor}(N,P) = 0\) so the map induced by \(P \to M\) is 0.

Notice that if \(F_0\) is fixed, then there is at most one universal \(\delta\)-functor (up to a unique isomorphism) \(F\) extending \(F_0\).
Lecture 10: Flat Modules - 02/14/2020

What we saw last time is that, modulo some details left to the reader, if $F$ is coeffaceable, then $F$ is a universal $\delta$-functor. This means that in particular, if $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to cB$ is additive, then $L_nF$ is a coeffaceable $\delta$-functor, hence universal. From this we get that

$$L_n(N \otimes (\_)) : \text{R-mod} \to \text{Ab}$$

and

$$R^n \text{Hom}(N, (\_)) : \text{R-mod} \to \text{Ab}.$$  

We also have

$$L_n((\_) \otimes M) : \mathcal{A} \to \text{Ab}$$

and

$$R^n \text{Hom}(\_, M) : \mathcal{A} \to \text{Ab}.$$  

Note that the functor in the last example is contravariant and left-exact. So we obtain the derived functor by taking a projective resolution in $\mathcal{A}$. We will show that there are natural isomorphisms between the corresponding derived functors.

**Lemma 7.** If $P$ is projective then $P \otimes (\_)$ is an exact functor. We say, in this case, $P$ is flat.

**Proof.** If $P$ is free then $P \cong R^0$. Assume

$$0 \to N' \to N \to N'' \to 0$$

is exact. Then

$$0 \to P \otimes N' \to P \otimes N \to P \otimes N'' \to 0.$$ 

Since, $P \otimes N = N^0$ and the maps in this second sequence are just the corresponding direct sums, this second sequence is also exact. In general, if $P$ is projective then all we have is that there exists a free module $F$ such that $F = P \oplus Q$ for some module $Q$. Then we have the following diagram, where the vertical maps are induced by the inclusion $P \hookrightarrow F$ and the identity $N \to N$.

$$
\begin{array}{cccccc}
0 & \longrightarrow & P \otimes N' & \longrightarrow & P \otimes N & \longrightarrow & P \otimes N'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F \otimes N' & \longrightarrow & F \otimes N & \longrightarrow & F \otimes N'' & \longrightarrow & 0
\end{array}
$$

□

**Theorem 6.**

$$L_n(N \otimes (\_))(M) = L_n((\_) \otimes M)(N)$$

$$R^n \text{Hom}(N, (\_)) = R^n \text{Hom}(\_, M)(N)$$

By equality we mean that there is a natural isomorphism.

**Proof.** We will show that $L_n(N \otimes (\_))(M)$ is a universal $\delta$-functor of $N$. Since $L_n((\_) \otimes M)(N)$ is a universal $\delta$-functor of $N$ and $L_0((\_) \otimes M)(N) = N \otimes M = L_0(N \otimes (\_))(M)$, once we prove this we will have maps induced in both directions by universality and so they must be isomorphisms. Suppose $P$ is flat, and choose a projective resolution $Q \to M$. Then

$$L_n(P \otimes (\_))(M) = \text{H}_n(P \otimes Q) = \begin{cases} P \otimes M, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$
It follows that all flat modules coefface the functor, so we are done. We leave the proof of the second isomorphism as an exercise. 

Just so we have an idea of what flat modules look like, at least in the category $\text{Ab}$, we state the following proposition.

**Proposition 9.** Flat abelian groups are torsion free abelian groups.

In general, we have the following hierarchy

$$\text{free} \Rightarrow \text{projective} \Rightarrow \text{flat}.$$ 

There are some conditions in which inclusions go both ways. For example, we have seen that if $R$ is a PID, then a module is projective if and only if it is free. We also have the following interesting theorem of Lazard.

**Theorem 7.** All flat modules are filtered colimits of free modules.

Our next step is to obtain as much intuition as we can about what the functors $\text{Tor}_n(N,M)$ and $\text{Ext}_n(N,M)$ are telling.

**Definition 16.** If $M,N \in \mathcal{A}$, an abelian category, an extension of $M$ by $N$ is an exact sequence

$$0 \rightarrow N \rightarrow X \rightarrow M \rightarrow 0.$$ 

Turns out that extensions for fixed $M$ and $N$ form a category where the morphisms are given by commutative diagrams like the following

$$
\begin{array}{cccccc}
0 & \rightarrow & N & \rightarrow & X & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \parallel & & \parallel & & \\
0 & \rightarrow & N & \rightarrow & Y & \rightarrow & M & \rightarrow & 0
\end{array}
$$

Notice that if $X = Y = N \oplus M$ then the options for the vertical map in the center are limited, namely we must have it be

$$
\begin{pmatrix}
1 & 0 \\
f & 1
\end{pmatrix}
$$

for some $f \in \text{Hom}(M,N)$.

**Definition 17.** $\text{Ext}^1(M,N)$ is the set of extensions of $M$ by $N$ modulo equivalence.

**Theorem 8.** For all $M$ and $N$, there is a natural isomorphism

$$\text{Ext}^1(M,N) \cong \text{Ext}^1(M,N)$$

**Exercise 31.** Weibel 3.2.2 Solution

**Exercise 32.** Weibel 3.2.3 Solution

**Exercise 33.** Weibel 3.2.4 Solution

**Exercise 34.** Weibel 3.3.1 Solution

**Exercise 35.** Weibel 3.3.2 Solution
The interpretations of Tor and Ext are not crucial for our voyage into homological algebra. However, these interpretations illustrate the use of homological algebra in mathematics quite well. Additionally, they are also quite fun. As such, we will spend some time continuing our discussion from last time and discuss the meaning of these particular derived functors. We will start with consequences of Theorem 8 stated last time. First we note that Ext\(_1(M,N)\) has a lot of structure. So Theorem 8 tells us that all of this structure is transferred to the set of extensions. In particular, Ext\(_1(M,N)\) is a covariant functor in \(N\) and contravariant in \(M\). Therefore, if \(M' \rightarrow M\) is a morphism of \(R\)-modules, then we should have a map

\[
\text{Ext}_1(M,N) \rightarrow \text{Ext}_1(M',N).
\]

This map is illustrated in the following diagram, where the right-hand square is a pullback.

\[
\begin{array}{ccc}
0 & \rightarrow & N \\
\downarrow & & \downarrow \\
0 & \rightarrow & N \\
\end{array}
\begin{array}{ccc}
& X' = M' \times_M X & M' \\
\downarrow & \downarrow & \downarrow \\
& X & M \\
\end{array}
\]

This is indeed functorial since pullbacks respect composition. Similarly, if \(N \rightarrow N'\) is a morphism of \(R\)-modules, then we get a map on Ext\(_1\) by the following diagram in which the left-hand square is a pushout.

\[
\begin{array}{ccc}
0 & \rightarrow & N \\
\downarrow & & \downarrow \\
0 & \rightarrow & N' \\
\end{array}
\begin{array}{ccc}
& X & M \\
\downarrow & \downarrow & \downarrow \\
& X' & M \\
\end{array}
\]

Furthermore, the square below commutes.

\[
\begin{array}{ccc}
\text{Ext}_1(M,N) & \rightarrow & \text{Ext}_1(M,N') \\
\downarrow & & \downarrow \\
\text{Ext}_1(M',N) & \rightarrow & \text{Ext}_1(M',N')
\end{array}
\]

We also should have that Ext\(_1(M,N)\) is an abelian group.

**Lemma 8.** There is a map

\[
\text{Ext}_1(M,N) \oplus \text{Ext}_1(M',N') \rightarrow \text{Ext}_1(M \oplus M',N \oplus N')
\]

**Proof.** Take the direct sum! \(\square\)

Now let \(f \oplus g\) denote the map

\[
\begin{pmatrix}
f & 0 \\
0 & g
\end{pmatrix}: M \oplus M \rightarrow N \oplus N
\]

define of two morphisms \(f,g: M \rightarrow N\). Let \(X\) and \(Y\) be two extensions of \(M\) by \(N\). Define

\[
X + Y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* (X \oplus Y)
\]

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The zero in $\text{Ext}^1(M, N)$ is the split extension
\[
\begin{array}{c}
0 \rightarrow N \\
\downarrow \\
0 \rightarrow M
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
N \oplus M \\
\begin{pmatrix} 0 & 1 \end{pmatrix}
\end{array}
\rightarrow
M \rightarrow 0.
\]

What else should we have? Well, for an exact sequence
\[
0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0
\]
we should have a map
\[
\text{Hom}(M, N'') \rightarrow \text{Ext}^1(M, N').
\]
This, again, is given by taking a pullback of a map $f \in \text{Hom}(M, N'')$ along $N \rightarrow N''$. The exercise is now to verify exactness of the induced long exact sequence in $\text{Ext}^1$. One can check that $\text{Ext}^1(M, N) = 0$ if $M$ is projective or $N$ is injective, implying that $\text{Ext}^1(\_\_, \_\_)$ is effaceable and coeefaceable in the appropriate slots so that
\[
\text{Ext}^1(\_\_, \_\_) = R^1 \text{Hom}(\_\_, \_) = \text{Ext}^1(\_\_, \_\_).
\]

**Exercise 36.** Show that $\text{Ext}^1(M, N)$ is effaceable in both $M$ and $N$. **Solution**

One can use this to define $\text{Ext}^n(M, N) = \text{Ext}^{n-1}(M, N')$ and try to find the meaning of the higher Ext-groups.

**Definition 18.** (Yoneda Ext.)

\[
\text{Ext}^n(M, N) = \{0 \rightarrow N \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow M \rightarrow 0 \mid \text{exact } \}/\sim
\]

Given $X, Y \in \text{Ext}^2(M, N)$, the morphisms are given by *butterflies*

\[
\begin{array}{c}
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
N \\
\rightarrow X_1 \\
\rightarrow X_2 \\
\rightarrow M \\
\rightarrow 0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Z \\
\downarrow \\
Y_1 \\
\rightarrow Y_2 \\
\rightarrow M \\
\rightarrow 0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\end{array}
\]

You can work out $\text{Ext}^3(M, N)$ yourself for fun, but the diagrams are a bit insane. Moving right along, remember that Tor is the left-derived functor of the tensor product. So Tor should behave similarly to the tensor product. In particular, it should satisfy a universal property similar to the tensor product.

**Definition 19.** Suppose $M, N, P$ are abelian groups. A *biextension* of $M, N$ by $P$ is a family of extensions
\[
0 \rightarrow P \rightarrow X(m) \rightarrow N \rightarrow 0
\]
for all $m \in M$. These come with isomorphisms
\[
X(m + m') \simeq X(m) + X(m'),
\]
that is, a biextension is a homomorphism $M \rightarrow \text{Ext}^1(N, P)$.

So for $A$ an abelian group and an injective $P$,

\[
\text{Hom}(\text{Tor}_A(M, N), P) = \text{Biext}(M, N, P).
\]
Today we will introduce limits and colimits. In essence, these are two different ways of forming a new object out of a collection of objects with some relations (morphisms) between them. The particularly nice thing about these constructions is that they are characterized by universal properties. After introducing these and seeing some examples we will be interested in how these behave with respect to the derived functors Tor and Ext. Long story short, they respect one another. First let us recall the definitions.

**Definition 20.** An *I*-shaped diagram in $\mathcal{C}$ is functor from a small category $I$ into $\mathcal{C}$, $X : I \to \mathcal{C}$.

For each $\xi \in \mathcal{C}$ there is a constant diagram $\Delta(\xi)$ which is just the constant functor sending all objects of $I$ to $\xi$ and all morphisms to the identity $id_\xi$. Now consider the set of all natural transformations from an arbitrary diagram $X$ to $\Delta(\xi)$, denoted $\text{Hom}(X, \Delta(\xi))$.

**Definition 21.** A colimit of a diagram $X : I \to \mathcal{C}$ is an object $\text{colim}X$ together with a natural transformation $\eta : X \Rightarrow \Delta(\text{colim}X)$ that is universal. A limit of a diagram $X : I \to \mathcal{C}$ is an object $\text{lim}X$ together with a natural transformation $\alpha : \Delta(\text{lim}X) \Rightarrow X$ that is universal.

**Remark 3.** In the above definition, we used an underline to denote the object underlying the limit and colimit to distinguish from the total data of these constructions which includes a natural transformation. This underline will be dropped in what follows for convenience. It should, however, be evident from context whether we are referring to the object or the total data.

By universal we mean, for example if $\xi$ is another object of $\mathcal{C}$ with a natural transformation $\eta' : X \Rightarrow \Delta(\xi)$ then there is a morphism $\text{colim}X \to \xi$ which when viewed as a natural transformation between the corresponding constant functors gives a factorization of $\eta'$ in terms of $\eta$. We have already seen many examples of these. Products and coproducts are limits and colimits of the diagrams with no morphisms. Pullbacks and pushouts are limits and colimits of the diagrams

$$A \longrightarrow B \leftarrow C$$

and

$$A \leftarrow B \longrightarrow C$$

respectively. Kernels and cokernels are also examples. Explicitly, the kernel and cokernel of a map $f : A \to Y$ are the limit and colimit, respectively, of the diagram below.

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \\
0 & \xrightarrow{0} & B
\end{array}$$

Such limits and colimits are called *equalizers* and *coequalizers*. Yet another nice thing about abelian categories is that limits and colimits have explicit constructions as kernels and cokernels of particular maps. Explicitly, we have

$$\text{lim}X = \ker \left( \prod_{i \in I} X(i) \to \prod_{q : i \to j} X(j) \right)$$

where the map is given by $(a_i) \mapsto (q(a_i) - a_j)_{q : i \to j}$. Similarly,

$$\text{colim}X = \text{coker} \left( \oplus_{q : i \to j} X(i) \to \oplus_{j \in I} X(j) \right)$$

where the map is given by $a_i[q] \mapsto q(a_i) - a_i$. This implies a nice result.
Corollary 1. Abelian categories have finite limits and colimits.

Example 11. Let \( p \in \mathbb{Z} \) be a prime. Then the \( p \)-adic integers are defined as the colimit
\[
\hat{\mathbb{Z}}_p = \text{colim}(\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \mathbb{Z}/p^3\mathbb{Z} \leftarrow \cdots).
\]
A category, \( \mathcal{A} \), for which all limits (resp. colimits) exist is called \textit{complete} (resp. \textit{cocomplete}). In this case, we can view the limit and colimit as functors
\[
\text{lim, colim} : \mathcal{A}^I \to \mathcal{A},
\]
where \( \mathcal{A}^I \) denotes the functor category.

Proposition 10. Let \( I \) be a small category such that all functors \( F : I \to \mathcal{A} \) have limits and colimits. Then the functor \( \text{lim} \) is right adjoint to the diagonal functor \( \Delta : \mathcal{A} \to \mathcal{A}^I \) sending each object \( \xi \in \mathcal{A} \) to the constant functor \( \Delta(\xi) \), and \( \text{colim} \) is left adjoint to \( \Delta \).

Proof. We want to show that there is a natural isomorphism \( \text{Hom}_{\mathcal{A}^I}(\Delta(\xi), F) \cong \text{Hom}_\mathcal{A}(\xi, \text{lim} F) \). Given a natural transformation \( \Delta(\xi) = \Rightarrow F \), the universal property of the limit gives rise to a morphism \( \xi \to \text{lim} F \). It is evident that any morphism \( \xi \to \text{lim} F \) induces a natural transformation \( \Delta(\xi) = \Rightarrow \Delta(\text{lim} F) \) which one can compose with the universal natural transformation \( \Delta(\text{lim} F) = \Rightarrow F \) to obtain \( \Delta(\xi) = \Rightarrow F \). This isomorphism is natural in both \( \xi \) and \( F \) since the construction used only natural transformations. The proof of the statement for colimits is nearly identical. \( \square \)

Theorem 2.6.10 in Weibel states that left adjoints preserve colimits and right adjoints preserve limits. Thus, we have that limits commute with limits and colimits commute with colimits.

Example 12. Let’s consider \( \text{Hom}(\mathbb{Z}[\frac{1}{p}], \mathbb{Q}/\mathbb{Z}) \). To determine a homomorphism in this set we must choose where 1 goes. The set of such choices is \( \mathbb{Q}/\mathbb{Z} \). Now we must choose where \( \frac{1}{p} \) goes, but this is restricted to lie in \( \mathbb{Q}/p\mathbb{Z} \) depending on the first choice. Similarly, \( \frac{1}{p^2} \) has a choice set \( \mathbb{Q}/p^2\mathbb{Z} \) and so we can compute
\[
\text{Hom}(\mathbb{Z}[\frac{1}{p}], \mathbb{Q}/\mathbb{Z}) = \text{colim}(\mathbb{Q}/\mathbb{Z} \leftarrow \mathbb{Q}/p\mathbb{Z} \leftarrow \mathbb{Q}/p^2\mathbb{Z} \leftarrow \cdots) = \hat{\mathbb{Q}}_p.
\]
This is used in one of the calculations in Weibel chapter 3 to compute \( \text{Ext}^1(\mathbb{Z}[p^{-1}]/\mathbb{Z}, \mathbb{Z}) \), which was also used in one of the exercises. We end this lecture by introducing particularly interesting limits and colimits. Let \( I \) be the poset category whose objects are the natural numbers 0, 1, 2..., and morphisms \( n \to m \) are given by relations \( n \leq m \). A functor \( F : I \to \mathcal{A} \) is then a sequence of objects and morphisms in \( \mathcal{A} \):
\[
A_0 \to A_1 \to A_2 \to \cdots,
\]
while a contravariant functor \( F : I^\text{op} \to \mathcal{A} \) is a tower:
\[
\cdots \to A_2 \to A_1 \to A_0.
\]
The colimit in the former is called the \textit{direct limit}, is denoted \( \lim_{\to} A_i \). The limit in the latter is called the \textit{inverse limit}, and is denote \( \lim_{\to} A_i \).

Exercise 37. Weibel 3.5.1 Solution
Exercise 38. Weibel 3.5.2 Solution
Exercise 39. Weibel 3.5.3 Solution
Exercise 40. Weibel 3.5.5 Solution

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As mentioned last time, under suitable circumstances like \( \mathcal{A} \) being complete and cocomplete, we can view limits and colimits as functors \( \mathcal{A}^I \to \mathcal{A} \) for any small category \( I \).

**Proposition 11.** Suppose \( I \) is a small category and \( \mathcal{A} \) is an abelian category with enough projectives and all direct sums (i.e. \( \mathcal{A} \) is complete). Then \( \mathcal{A}^I \) has enough projectives. (Note: a projective in \( \mathcal{A}^I \) is a functor \( F \) such that \( F(i) \) is projective for all objects \( i \) of \( I \)).

**Proof.** For each object \( i \) in \( I \), we have an exact evaluation functor

\[
e_i : \mathcal{A}^I \to \mathcal{A}
\]

sending a functor \( P \) to the object \( P(i) \). This is right adjoint to the functor \( F_k : \mathcal{A} \to \mathcal{A}^I \) defined by

\[
F_k(Q) = \{ i \mapsto \oplus \text{Hom}(i,k)Q \}.
\]

Morphisms are induced from the morphisms \( \text{Hom}(i',k) \to \text{Hom}(i,k) \) for \( i \to i' \). Explicitly, if \( \varphi : i \to i' \) in \( I \), then the coordinate projection \( F_k(Q) \to Q \) onto the factor corresponding to \( \psi : i' \to k \) is the projection of \( F_k(Q) \) onto to the factor corresponding to \( \psi \circ \varphi \). We leave the verification of this to the reader. We then have that \( F_k \) is left-adjoint to an exact functor so that \( F_k \) preserves projectives implying \( F_k(Q) \) is projective for any projective \( Q \) in \( \mathcal{A} \). Suppose \( M \in \mathcal{A}^I \). Let \( P_i \to M(i) \) be a an epi for each \( i \), with \( P_i \) projective. Then

\[
\oplus_i F_i(P_i) \to M
\]

is a surjection from a projective.

\[\square\]

Using the fact that the inverse limit and direct limit are left and right exact, respectively, we can then define the derived functors

\[
R^n \lim : \mathcal{A}^I \to \mathcal{A}
\]

and

\[
L^n \lim : \mathcal{A}^I \to \mathcal{A}.
\]

We will use the shorthand \( \lim^1 \) for \( R^1 \lim \). We will focus on understanding the right derived functors of the inverse limit functor. As we will see, these vanish for \( n > 1 \) so \( \lim^1 \) is the only interesting case. Throughout this section, let \( I \) denote the standard poset category of natural numbers. We will consider the case \( \mathcal{A} = \text{Ab} \). In this case, \( \lim^1 \) has a particularly nice description.

**Definition 22.** Let \( M \in \mathcal{A}^I \) and take \( \mathcal{A} = \text{Ab} \). An \( \mathcal{M} \)-torsor is a functor \( P : I \to \text{Set} \) such that \( M(i) \) acts on \( P(i) \neq \emptyset \) simply transitively and the actions commute with the maps in \( I \).

A natural transformation of functors that commutes with \( M(i) \) action for all objects \( i \) in \( I \) is a morphism of \( M \)-torsors. So these form a category, in fact a groupoid (all morphisms are isomorphisms). We claim that \( \lim^1 M \) is in fact the set of \( M \)-torsors modulo isomorphism. In fact, let us make the following definition, and prove that \( \lim^{(1)} \) and \( \lim^1 \) are the same.

**Definition 23.** If \( M : I \to \text{Ab} \) is a functor, define \( \lim^{(1)}(M(i)) \) be the set of \( M \)-torsors modulo isomorphism.

We note that \( M \) itself can be considered an \( M \)-torsor by composing with the forgetful functor \( \text{Ab} \to \text{Set} \) and letting \( M(i) \) act on its underlying set by multiplication for each \( i \). If \( P \) is an \( M \)-torsor, \( P \equiv M \) if and only if \( \lim P(i) \neq \emptyset \).
Example 13. Let $I = 0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots$ and $M(n) = p^n \mathbb{Z}, N(n) = \mathbb{Z}/p^n \mathbb{Z}$. Then $\varprojlim M(n) = 0$, and we have the following Exact sequence where the index $n$ is suppressed.

$$0 \rightarrow M \rightarrow \mathbb{Z} \rightarrow N \rightarrow 0.$$ 

Then for every $\alpha \in \varprojlim N(n) = \hat{\mathbb{Z}}_p$ we get a coset of $M$ in $\mathbb{Z}$. The system $P(n) = \pi^{-1}(\alpha_n)$ is an $M$-torsor.

This construction in the example is actually a special case of something more general, namely, the existence of a connecting homomorphism in the following proposition.

Proposition 12. Let

$$0 \rightarrow M' \rightarrow M \rightarrow M''$$

be exact. Then there is a long exact sequence

$$0 \longrightarrow \varprojlim M' \longrightarrow \varprojlim M \longrightarrow \varprojlim M'' \overset{\delta}{\longrightarrow} \varprojlim (1) M' \longrightarrow \varprojlim (1) M \longrightarrow \varprojlim (1) M''$$

Proof. TODO: the proof. □

<+++>

Proposition 13. $\varprojlim (1)$ is effaceable.

Proof. TODO: the proof. □

<+++>

This completes the proof that $\varprojlim^1 = \varprojlim^{(1)}$. We will end today’s lecture by showing explicitly that $\varprojlim^{(1)}$ satisfies a particularly nice property that $\varprojlim^1$ satisfies.

Definition 24. (Mittag-Leffler Condition) We say that $M : I \rightarrow \mathcal{A}$ satisfies the **Mittag-Leffler condition** if the images $M(n + k) \rightarrow M(n)$ stabilize for large enough $k$.

This condition ends up being very useful as it is a sufficient condition to deduce that $\varprojlim^1 M(i) = 0$. The reader can see Weibel’s text for a proof of this general result. Here we prove explicitly that this is also the case of $\varprojlim^{(1)}$.

Theorem 9. If $M$ satisfies the Mittag-Leffler condition then $\varprojlim^{(1)} M = \{M\}$

Proof. Let $P$ be an $M$-torsor. Let

$$P'(n) = \bigcap_{k \geq 0} \text{im}(P(n + k) \rightarrow P(n)).$$

This is nonempty by assumption. Then $P'(n + k) \rightarrow P'(n)$ is surjective for all $k \geq 0$. So $\varprojlim P'(n) \subset \varprojlim^1 P(n)$ is nonempty. Thus, by our note above, $\varprojlim^1 P(n) \neq \emptyset$ implying $P \cong M$. □
Lecture 14: Filtered Colimits - 02/28/2020

We will finish our discussion on limits and then we will move on to spectral sequences.

**Definition 25.** A (small) category $I$ is called **filtered** if

1. $\forall i,j \in I, \exists k \in I$ with morphisms $i \to k$ and $j \to k$ (upper bound)
2. $\forall i,j \in I$ with morphisms $\alpha : i \to j$ and $\beta : i \to j, \exists$ a coequalizer $\gamma : j \to k$.

Thinking loosely in terms of homotopy theory, this definition states that $I$ is, in some sense, **contractible**. Colimits of functors with domain a filtered category are called filtered colimits and usually denoted the same as a direct limit $\lim$. 

**Theorem 10.** Filtered colimits commute with finite limits in the category of sets.

Consider $X : I \times J \to \text{Set}$, with $I$ finite and $J$ filtered. The theorem above states that the canonical map

$$\lim_j \lim_i X(i,j) \longrightarrow \lim_i \lim_j X(i,j)$$

obtained from the universal properties of the limits involved is an isomorphism.

**Proof:** All (finite) limits are built from equalizers and finite products. It follows that it suffices to prove the theorem for $I = \{ 0 \}$ and $I = \{ 0 \to 1 \}$. We will prove this for the latter case and leave the former as an exercise. Let $\alpha, \beta$ be the two nonidentity morphisms in $I$. Note

$$\lim_i X(i,j) = \{ x_0 \in X(0,j) | \alpha(x_0) = \beta(x_0) \}.$$ 

In the colimit, $x_0 \in X(0,j)$ and $x_0' \in X(0,j')$ are equivalent if there exists morphisms $j \to k$ and $j' \to k$ that map $x_0, x_0'$ to some $x_k'$ under $X$. Thus, we have a map

$$\lim_i \lim_j X(i,j) \to \lim_i \lim_j X(i,j) = \{ [x_0] \in \lim_j X(0,j) | \alpha(x_0) = \beta(x_0) \},$$

mapping the image of $x_0$ in the colimit $\lim_i \lim_j$ to the equivalence class $[x_0]$. We first show that this is surjective. Suppose $[x_0] \in \lim_i \lim_j X(0,j)$ and $[\alpha(x_0)] = [\beta(x_0)]$. then $x_0 \in X(0,j)$ for some $j$. IF $\alpha(x_0) \neq \beta(x_0)$, we still know that $\exists \gamma : j \to j'$ such that $\gamma(\alpha(x_0)) = \gamma(\beta(x_0))$ because $[\alpha(x_0)] = [\beta(x_0)]$. So $\gamma(x_0) \in \lim_j X(i,j')$ implying $[\gamma(x_0)] \in \lim_j \lim_i X(i,j)$ is such that $\phi([\gamma(x_0)]) = [x_0]$. 

**Corollary 2.** Filtered colimits are exact.

**Proof:** Colimits commute with colimits so colimits are right exact. Since kernels are finite limits, filtered colimits preserve kernels and cokernels.

**Corollary 3.** Homology commutes with filtered colimits, that is

$$H_n(\lim_i C_i) = \lim_i H_n(C_i).$$

**Proposition 14.** $\text{Tor}$ commutes with filtered colimits. Explicitly, $M : I \to R\text{-mod}$ where $I$ is filtered, and $N$ an $R$-module,

$$\text{Tor}_n(\lim_i M(i), N) \cong \lim_i \text{Tor}_n(M(i), N).$$
Proof. Choose a projective resolution \( P \to N \). Then
\[
\text{Tor}_n(\lim M(i), N) = H_n(\lim M(i) \otimes N) = \lim H_n(M(i) \otimes N) = \lim \text{Tor}_n(M(i), N).
\]

Corollary 4. Let \( M \) be an abelian group. \( M \) is Tor-acyclic if and only if \( M \) is torsion-free.

Proof. For any \( M, M = \bigcup_i M_i = \lim_i M_i \), where \( M_i \) are the finitely generated subgroups of \( M \). Thus
\[
\text{Tor}_n(M, -) = \lim_n \text{Tor}_n(M_i, -).
\]
If \( M \) is torsion-free then so are the \( M_i \) implying \( M_i \) are free, since they are finitely generated. Thus
\[
\text{Tor}_n(M, -) = 0, \quad n > 0.
\]

Recall that an \( R \)-module is called \emph{finitely presented} if \( \exists \) an exact sequence
\[
R^m \to R^n \to M \to 0,
\]
where \( m \) and \( n \) are finite.

Proposition 15. \( M \) is finitely presented if and only if
\[
\text{Hom}(M, \lim_i N_i) \equiv \lim_i \text{Hom}(M, N_i).
\]
We leave the proof of this to the reader. However, a nice corollary to this is that if \( M \) is finitely presented and \( R^n \to M \) is surjective, then the kernel is finitely generated.

\textbf{Lecture 15: Spectral Sequences - 03/02/2020}

Alright, it's now time to talk about spectral sequences. Let us start by describing the situation where one usually meets spectral sequences. Suppose you have a double complex \( C_{pq} \) displayed visually below.

\[
\vdots \quad \vdots \quad \vdots \\
\cdots \to C_{22} \to C_{12} \to C_{02} \to \cdots \\
\cdots \to C_{21} \to C_{11} \to C_{01} \to \cdots \\
\cdots \to C_{20} \to C_{10} \to C_{00} \to \cdots \\
\vdots \quad \vdots \quad \vdots 
\]
Where would one encounter such a complex in the first place? One important situation is when you try to compute a composition of derived functors. These are each computed separately by taking the homology/cohomology of an injective or projective resolution. When we compose them, we then have a double complex to compute the homology/cohomology of. We can combine the data of a double complex into an ordinary chain complex

$$\text{Tot}(C.)_n = \sum_{p+q=n} C_{pq},$$

where the differential is $d = d^v + d^h$, where $d^v$ is the vertical differential and $d^h$ the horizontal differential. So we are interested in computing the homology of this double complex, which amounts to computing the homology of $\text{Tot}(C.)$. One place to start would be to compute the vertical homology first. With this in mind, set $E^1_{pq} = H_q(C_p)$. We then have the following sequence of horizontal complexes in which the differentials are induced by $d^h$.

\[
\cdots \rightarrow E^1_{22} \rightarrow E^1_{12} \rightarrow E^1_{02} \rightarrow \cdots \\
\cdots \rightarrow E^1_{21} \rightarrow E^1_{11} \rightarrow E^1_{01} \rightarrow \cdots \\
\cdots \rightarrow E^1_{20} \rightarrow E^1_{10} \rightarrow E^1_{00} \rightarrow \cdots \\
\vdots
\]

We can then take the horizontal homology of this resulting sequence of complexes. With this in mind, set $E^2_{pq} = H_p(E^1_q) = H_p(H_q(C_\cdot))$. We then have the following lattice of homology groups.

\[
\vdots \\
E^2_{22} \quad E^2_{12} \quad E^2_{02} \\
\cdots \quad E^2_{21} \quad E^2_{11} \quad E^2_{01} \quad \cdots \\
E^2_{20} \quad E^2_{10} \quad E^2_{00} \\
\vdots
\]
Now let us see how close this got us to our goal of computing the homology of $\text{Tot}(C..)$. For simplicity, suppose that $C.$ is a first quadrant double complex, meaning $C_{p,q} = 0$ if $q$ or $p$ are less than zero. Then at least indegree zero, we have succeeded:

$$H_0 H_{00} = \frac{C_{00}}{d^{\nu}(C_{01}) + d^{h}(C_{10})} = H_0 \text{Tot}(C.).$$

However, in degree one we still have some work to do. First, what should we expect to get? Well a rough description of this is the following

$$H_1 \text{Tot}(C.) = \frac{\{(x_{01},x_{10}) | d^{\nu}(x_{01}) = d^{h}(x_{10})\}}{\langle (d^{\nu}(x_{02}),0),(d^{h}(x_{11}),d^{\nu}(x_{11})),(0,d^{h}(x_{02})) \rangle}.$$

Now let’s see what we get by following our procedure above. First we get the vertical homology

$$H_1(C_0) = \frac{\{x_{01} | d^{\nu}(x_{01}) = 0\}}{\{x_{01} | \exists x_{02} \in C_{02}, d^{\nu}(x_{02}) = x_{01}\}}.$$

Then we take the horizontal homology, this time at degree zero so that the total degree is 1:

$$H_0 H_1 = \frac{\{(x_{01},0) | d^{\nu}(x_{01}) = 0\}}{\{(d(x_{02}),0), | x_{02} \in C_{02} \} + \{(d^{h}(x_{11}),0), | d^{\nu}(x_{11}) = 0\}}.$$

It follows that we have a map $H_0 H_1 \to H_1 \text{Tot}(C.).$. This is not quite an isomorphism, but there is an exact sequence

$$\cdots \to H_0 H_1 \to H_1 \text{Tot}(C.) \to H_1 H_0 \to 0.$$

So we were able to fit what we wanted to compute into a long exact sequence consisting of simpler things to compute. This is, in essence, the utility of spectral sequences. We introduce the formal definition of a spectral sequence now so that the reader is aware of the kind of objects we are working with.

**Definition 26.** A (homology) **spectral sequence** (starting with $E^a$) in an abelian category $\mathcal{A}$ consists of the following data:

1. A family $(E^r_{pq})$ of objects of $\mathcal{A}$ defined for all integers $p,q,$ and $r \geq a$.
2. Morphisms $d^r_{pq}: E^r_{pq} \to E^r_{p-r,q+r-1}$ called differentials, which satisfy $d^r d^r = 0$.
3. Isomorphisms between $E^r_{pq}$ and the homology of $E^r$ at the $(p,q)$-spot.

We call $E^a$ the ‘$E^a$-page’. Let us move on to re-introduce objects for which spectral sequences are a particularly useful tool.

**Definition 27.** A **filtration** of an object of an abelian category, $M \in \mathcal{A}$ is a sequence of subobjects

$$\cdots \subset F_n M \subset F_{n+1} M \subset \cdots$$

(Note: in a general abelian category, subobjects of $M$ are equivalence classes of objects with injections into $M.$) These define a metric on $M$

$$\log \delta(x,y) = \min\{n | x - y \in F_n M\}.$$

Define the **associated graded** by

$$\text{gr} M = \bigoplus_{n} \frac{F_n M}{F_{n-1} M}.$$
Example 14.

\[ \cdots \subset p^2\Z \subset p\Z \subset \Z. \]

The metric completion there is \( \hat{\Z}_p \). The \( n \)-th associated graded of this filtration is \( \gr_n = p^{-n}\Z / p^{-(n-1)}\Z \cong \Z / p\Z \). The total associated graded is \( \gr = \F_p[t] \).

We can use filtrations do obtain a double complex and hence a spectral sequence. We should note, however, that in this context a spectral sequence does not necessarily compute the homology for us, but it does give us information about how the homology changes as we move along the filtration. Often this combined with some additional information we have about the objects in question will be enough to compute the desired homology.

If \( C \) is a complex of filtered objects, then the homology gets an induced filtration:

\[ F_p H_n C = \frac{F_p C_n \cap Z_n C}{F_p C_n \cap B_n C}. \]

We want to compute the associated graded, which is given in degree \( p \) by

\[ \gr_p H_n C = \frac{F_p C_n \cap Z_n C}{F_p C_n \cap B_n C + F_{p-1} C_n \cap Z_n C}. \]

We can do this by noting that this is in fact just the \( n \)-th homology of the following sequence

\[ d^{-1}(F_p C_n) \xrightarrow{F_p C_n} \frac{F_p C_n}{F_{p-1} C_n} \xrightarrow{d(F_p-1 C_n)} C_{n-1}. \]

**Exercise 41.** Weibel 5.2.1 Solution

**Exercise 42.** Weibel 5.2.2 Solution

**Exercise 43.** Weibel 5.1.1 Solution

**LECTURE 16: SPECTRAL SEQUENCES VIA FILTRATIONS - 02/06/2020**

**Definition 28.** A **filtered complex** is a filtered object in the category of chain complexes. This is the same as having a filtration \( FC_n \) for each \( n \).

We emphasize that this implies \( d(F_p C_n) \subset F_p C_{n-1} \). The idea is that we can use the filtrations on \( C_{n-1}, C_n, C_{n+1} \) to obtain information about \( H_n(C) \). The first thing to notice in this direction is that the filtration on \( C \) induces a filtration on \( H_n(C) \), which we saw last time:

\[ F_p H_n C = \frac{F_p C_n \cap Z_n C}{F_p C_n \cap B_n C}. \]

The associated graded \( \gr H_p C \) is then given by

\[ \gr_p H_n C = \frac{F_p H_n C}{F_{p-1} H_n C}. \]

We introduce some new notation,

\[ \gr_{p,q} C_n = \frac{F_q C_n}{F_p C_n}, \]

\[ \text{Exercise 43. Weibel 5.1.1 Solution} \]
and similarly
\[ \text{gr}_{p,q} H_n C. = \frac{F_q H_n C.}{F_p H_n C.}. \]

Note in this notation \( p < q \). Next we use the filtration to approximate the kernel and the image
\[ H_n C. (r,s) = \frac{d^{-1}(F_s C_{n-1})}{d(F_r C_{n+1})} \]
if \( \cup_r F_r C_{n+1} = C_{n+1} \) and \( \cap_s F_s C_{n-1} = 0 \) then these approximate \( H_n C. \) as \( r,s \to \infty \).

**Remark 4.** Throughout this lecture we use the following notation. If \( A, B \subset C \), write
\[ A/B = A/(A \cap B) = (A + B)/B = \text{im}(A \to C/B). \]

Now let’s combine these two ideas.
\[ \text{gr}_{[p,q]} H_n (r,s) = \frac{F_q C_n \cap d^{-1}(F_s C_{n-1})}{F_p C_n + d(F_r C_{n+1})}. \]

What we are doing here is taking approximate cycles and modding out by approximate boundaries. Now what happens when we apply the differential to this object? Well It certainly will land in a quotient of \( F_s C_{n-1} \). Further, what we quotient by must contain \( d(F_p C_n) \). Aside from this we have freedom to choose what this quotient is. In particular, we may choose the image so that \( d \) maps into another one of our grading quotients as shown below to the right. It turns out we can extend this in the opposite direction now, and consider the map induced by the differential into \( \text{gr}_{[p,q]} H_n (r,s) \). This gives us the map on the left.

\[ \text{gr}_{[p',q']} H_n (p',q') \xrightarrow{d} \text{gr}_{[p,q]} H_n (r,s) \xrightarrow{d} \text{gr}_{[s',r']} H_{n-1} (p,q') \]

We note that here \( q', p', s', r' \) are arbitrarily chosen. Evidently, we now obtain a chain complex, so we may take its homology and obtain
\[ \frac{F_q C_n \cap d^{-1} F_{s'}(C_{n-1})}{F_p C_n + d(F_{r'}(C_{n+1})}. \]

This is nothing but \( \text{gr}_{[p,q]} H_n (r',s') \). But now we have that \( r' < r \) and \( s' < s \) so this approximation is better than the one we started with! We can then continue in the same manner and improve our approximation once more. The important takeaway here is that everytime we take the homology of the current complex we get a better approximation of the associated graded of \( H_n \). We note that \( p' \) and \( q' \) played no significant role here. They will be important later when we extend these to longer sequences.

Now assume that \( C \) is a filtered complex. We then have
\[
\begin{array}{c}
F_{p-1} C_{n+1} \xrightarrow{d} F_{p-1} C_n \xrightarrow{d} F_{p-1} C_{n-1} \\
| \downarrow \quad \downarrow \quad \downarrow \\
F_p C_{n+1} \xrightarrow{d} F_p C_n \xrightarrow{d} F_p C_{n-1} \\
| \downarrow \quad \downarrow \quad \downarrow \\
\cdots \xrightarrow{} \text{gr}_p C_{n+1} \xrightarrow{} \text{gr}_p C_n \xrightarrow{} \text{gr}_p C_{n-1} \xrightarrow{} \cdots
\end{array}
\]
But notice that the last row is the same things as

\[ \cdots \rightarrow \text{gr}_{p-1,p}\bar{H}_{n+1}(p-1,p) \rightarrow \text{gr}_{p-1,p}\bar{H}_n(p-1,p) \rightarrow \text{gr}_{p-1,p}\bar{H}_{n-1}(p-1,p) \rightarrow \cdots. \]

Let

\[ E'^{r}_{p,q} = \text{gr}_{p-1,p}\bar{H}_{p+q}(p-1+r,p-r). \]

Next time we will use this construction to study double complexes.

**Lecture 17: Spectral Sequence of a Double Complex - 03/09/2020**

Last time we constructed a spectral sequence for a filtered complex. Recall the construction of \( \text{gr}_{p,q}\bar{H}_{n}(r,s) \) as a quotient of 'almost cycles' by 'almost boundaries'. We also saw that if our original complex \( C \) is filtered, meaning \( d(F_p C) \subset F_p C \), then things become simpler and we get a nice spectral sequence with

\[ E'^{r}_{p,q} = \text{gr}_{p-1,p}\bar{H}_{p+q}(p-1+r,p-r). \]

Now we move on to a case of particular interest, namely a double complex, \( C \) of which a single anticommuting square is shown below.

\[
\begin{array}{ccc}
C_{pq} & \xrightarrow{d^h} & C_{p-1,q} \\
\downarrow{d^v} & & \downarrow{d^v} \\
C_{p,q-1} & \xrightarrow{d^h} & C_{p-1,q-1}
\end{array}
\]

Recall that there is a choice when defining what we mean by the total complex of this double complex

\[ \text{Tot}^\Pi(C.) = \prod_{p+q=n} C_{pq} \]

or

\[ \text{Tot}^\oplus(C.) = \oplus_{p+q=n} C_{pq}. \]

These are equal if there are only finitely many nonzero \( C_{pq} \) such that \( p + q = n \) for each \( n \). We define a filtration on this double complex by letting

\[ F_k C_{pq} = \begin{cases} C_{pq} & p \geq k \\ 0 & p < k \end{cases}. \]

Now to construct a sequence. The \( E^0 \)-page will simply be the double complex with differential given by \( d^v \). Notice that this now has the interpretation of being the associated graded of our filtered complex. The \( E^1 \)-page is obtained by taking the homology of the \( E^0 \) page and the differential will simply be induced by \( d^h \) on the quotients. This leads us to the very important \( E^2 \)-page.

Note that \( \text{gr}_p \bar{H}_n \) measures the difference between each step of the grading induced on homology. Then if \( F_{p} C_{n} = 0 \) for \( p << 0 \), \( F_{p} C_{n} = C - n \) for \( p >> 0 \) and all \( n \). Then

\[ \text{gr}_p \bar{H}_n(a,b) = \frac{d^{-1}(F_{k}C_{n-1}) \cap F_{p} C_{n}}{d(F_{k}C_{n+1}) + F_{p-1} C_{n}} = F_{p} \bar{H}_n C.. \]

This implies \( E'^{r}_{p,n-p} = \text{gr}_p \bar{H}_n C. \) for all \( r >> 0. \)

That is quite enough abstraction for now, let's move on to an example.
**Example 15.** Suppose $M, N$ are $R$-modules. Choose a projective resolution $P$ of $M$ and $Q$ of $N$. Then $P \otimes Q$ is a double complex. We note that the differentials may need to be adjusted in order to agree with our sign convention, but we ignore this detail for now. Let’s compute the homology using the spectral sequence we constructed. However, note that we made a choice above to take the filtration along the horizontal direction. Instead, we could start by filtering along the vertical direction. Let us see how these two approaches differ in this example. In the first case where we filter horizontally, our $E^0$-page looks like

\[
E^0: \begin{array}{ccc}
\vdots & \vdots & \vdots \\
P_2 \otimes Q_2 & P_1 \otimes Q_2 & P_0 \otimes Q_2 \\
\downarrow & \downarrow & \downarrow \\
P_2 \otimes Q_1 & P_1 \otimes Q_1 & P_0 \otimes Q_1 \\
\downarrow & \downarrow & \downarrow \\
P_2 \otimes Q_0 & P_1 \otimes Q_0 & P_0 \otimes Q_0
\end{array}
\]

From this, we see that the $E^1$-page, which we obtain by taking homology of $E^0$, is $E^1_{pq} = P_p \otimes H_q(Q)$. However, since $Q$ is a projective resolution, it is exact for $q \geq 1$. For $q = 0$ we have $H_0(Q) = B$. Visually, we have:

\[
E^1: \begin{array}{ccc}
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
\vdots & P_2 \otimes N & P_1 \otimes N & P_0 \otimes N
\end{array}
\]

Thus, taking homology gives us

\[
E^2_{pq} = \begin{cases} 
L_p(\_ \otimes N)(M) & q = 0 \\
0 & q \neq 0
\end{cases}.
\]

In the latter case where we filter vertically, we can follow a similar argument to obtain $E^1_{pq} = H_p(P) \otimes Q_q$, and

\[
E^2_{pq} = \begin{cases} 
L_q(M \otimes \_)(N) & p = 0 \\
0 & p \neq 0
\end{cases}.
\]

Since the complex we started with is a first-quadrant double complex, its filtration is canonically bounded and so it converges to the homology of the total complex in each case. This, in particular, proves that

\[
\text{Tor}(\_, N)(M) = L_0(\_ \otimes N)(M) = E^2_{00} = L_0(M \otimes \_)(N) = \text{Tor}(M, \_)(N).
\]

**Exercise 44.** Weibel 5.2.3 Solution

**Exercise 45.** Weibel 5.4.4 Solution

**Exercise 46.** Weibel 5.6.1 Solution

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This is our first zoom meeting! Let's see how it goes.

The problem we want to consider today is the following. Suppose

\[ \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \]

is a sequence of right exact functors. How do we relate \( L^n G, L^m F, L^p (G \circ F) \)? Choose a resolution \( P \) of \( X \in \mathcal{A} \). Then \( L^n F(X) = H_n F(P) \) and now we need to resolve \( F(P) \) by a double complex \( Q \). Then we can relate \( G(Q) \) to \( L^n G(L_m F(X)) \) and \( L^p (G F)(X) \).

**Theorem 11.** Suppose \( C \) is a chain complex in \( \mathcal{A} \) and that \( \mathcal{A} \) has enough projectives. There exists a double complex \( P \) such that the following holds:

1. Each \( P_m \) is a projective resolution of \( C_m \).
2. \( \text{im}(P_m \to P_{m-1}) \) is a projective resolution of \( \text{im}(C_m \to C_{m-1}) \).
3. \( H_m(P) \to H_m(C) \) is a projective resolution.
4. If \( C_m = 0 \) then \( P_{mn} = 0 \) for all \( n \).

**Proof.** The idea is to start with point 2. Arrange this by choosing a projective resolutions \( Q_m \to d(C_m) \) and \( R_m \to H_m(C) \). We will combine these two resolutions using the Horseshoe Lemma as follows.

\[
\begin{array}{cccccccc}
0 & \to & Q_{m+1} & \to & S_m & \to & R_m & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & d(C_{m+1}) & \to & Z_m(C) & \to & H_m(C) & \to & 0
\end{array}
\]

Since the sequence on the bottom is exact, the Horseshoe Lemma gives tells us that \( S_m = Q_{m+1} \oplus R_m \) is then a projective resolution of \( Z_m(C) \). We play the same game now in the form of the following diagram.

\[
\begin{array}{cccccccc}
0 & \to & S_m & \to & P_m & \to & Q_m & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & Z_m(C) & \to & C_m & \to & d(C_m) & \to & 0
\end{array}
\]

Now \( P_{mn} = S_m \oplus Q_m = Q_{m+1} \oplus R_m \oplus Q_m \) is a projective resolution of \( C_m \). We then have a double complex of the following form.

\[
\begin{array}{cccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
Q_{m+1} \oplus R_m \oplus Q_m & \to & Q_m \oplus R_{m-1} \oplus Q_{m-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
C_m & \xrightarrow{d} & C_{m-1}
\end{array}
\]
The horizontal map above is simply given, in matrix form, by the elementary $3 \times 3$ matrix $E_{13}$. We note that signs may be adjusted so that this forms a double complex in the sense of Weibel, i.e. in order for squares to anticommute.

The construction above is often called the **Cartan-Eilenberg resolution** of $C$. Now we can talk about the Grothendieck Spectral Sequence. This is a very powerful spectral sequence because many others that you will come across, such as the Leray spectral sequence, are special cases of this construction. The setting is the one we started with above. We have a composition of right-exact functors

$$
\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C},
$$

where $\mathcal{A}, \mathcal{B}$ have enough projectives. So we can compute the left-derived functors. Suppose $X \in \mathcal{A}$, and choose a projective resolution $P. \to X$ of $X$. Now choose a Cartan-Eilenberg resolution $Q. \to F(P.)$. Recall that this means that $Q_m. \to F(P_m)$ is a projective resolution. We now have a double complex $G(Q.)$ on which we can run our spectral sequence from last lecture. The $I^E_0$ page then looks like:

$I^E_0$:

$$
\begin{align*}
G(Q_{22}) & \to G(Q_{21}) \to G(Q_{20}) \\
G(Q_{12}) & \to G(Q_{11}) \to G(Q_{10}) \\
G(Q_{02}) & \to G(Q_{01}) \to G(Q_{00})
\end{align*}
$$

Now we make an additional assumption that $L_nG(P_m) = 0$ for all $n > 0$. An example of where this holds is if $F$ has an exact right-adjoint. Now the homology at the $p$-th slot of row $q$ is then $H_p(G(Q_q)) = L_pG(F(P))$, and these vanish for projective $P$, the $I^E_1$-page then looks like:

$I^E_1$:

$$
\begin{array}{ccc}
0 & 0 & G(P_2) \\
0 & 0 & G(P_1) \\
0 & 0 & G(P_0)
\end{array}
$$

Now let’s move on to the $I^E_2$-page. For this, we simply take the vertical homology above and get the following.
\[ I^2 E^2 : \]

\[
\begin{array}{ccc}
0 & 0 & L_2 GF(X) \\
0 & 0 & L_1 GF(X) \\
0 & 0 & L_0 GF(X)
\end{array}
\]

So we get that \( H_n(\text{Tot}(GQ)) = L_n GF(X) \).

Now let’s see what happens when we run our double-complex spectral sequence in the opposite direction. In this case,

\[ \text{II } E^0 : \]

\[
\begin{array}{ccc}
G(Q_{22}) & G(Q_{21}) & G(Q_{20}) \\
\downarrow & \downarrow & \downarrow \\
G(Q_{12}) & G(Q_{11}) & G(Q_{10}) \\
\downarrow & \downarrow & \downarrow \\
G(Q_{02}) & G(Q_{01}) & G(Q_{00})
\end{array}
\]

However, by our construction above, we know that \( Q_{\cdot,n} \) is a split complex, and \( G \), being an additive functor, preserves this structure. Thus, \( G(Q_{\cdot,n}) \) is a split complex, implying \( H_m(GQ) = G(H_mQ) \).

\textbf{How?} The \( E^1 \)-page is then:

\[ \text{II } E^1 : \]

\[
\begin{array}{ccc}
G(H_2(Q_2)) & \longrightarrow & G(H_2(Q_1)) & \longrightarrow & G(H_2(Q_0)) \\
G(H_1(Q_2)) & \longrightarrow & G(H_1(Q_1)) & \longrightarrow & G(H_1(Q_0)) \\
G(H_0(Q_2)) & \longrightarrow & G(H_0(Q_1)) & \longrightarrow & G(H_0(Q_0))
\end{array}
\]

Now recall that \( H_m(Q_{\cdot}) \to H_m(P_{\cdot}) = L_m F(X) \) is a projective resolution. So we get the following nice description of the \( E^2 \)-page.
To compute further, we need more information about the specific functors and categories that we are working with. In summary, we have created a spectral sequence of the form

\[ E^2_{pq} = L_p F(L_q G(X)) = \Rightarrow L_{p+q}(GF)(X), \]

and so we have succeeded in our goal of relating these derived functors. Next time we’ll see some examples of this construction.

Exercise 47. Weibel 5.6.2

Solution

LECTURE 19: DERIVED CATEGORIES - 03/16/2020

We are going to introduce derived categories today. Weibel takes the standard approach of motivating derived categories through algebraic topology. This is indeed a very important view and we refer the reader to the textbook for that view. Here we will try to motivate the idea with just homological algebra in mind. The overarching idea of what we have been doing in this class is use complexes to analyze objects of an abelian category. As we have seen, complexes, as well as beefed-up versions of complexes (e.g. double complexes, spectral sequences, ect.), are very useful tools. Let’s take the particular example of Grothendieck spectral sequences from last time.

Remark 5. Following Weibel and Grothendieck, we will now switch to talking about cochain complexes and cohomology.

Missed some of this argument.

The point is that we want to consider complexes equivalent when they are quasi-isomorphic. So we are in search of a category, \( D(\mathcal{A}) \), which is, in a sense, \( \text{Ch}(\mathcal{A}) \) modulo quasi-isomorphisms.

Proposition 16. If \( \mathcal{A} \) is an abelian category with enough injectives and \( C \) is a bounded below complex \( (C^n = 0 \forall n < 0) \) then there is a bounded below complex of injectives \( I \) and a quasi-isomorphism \( C \to I \).

Proof. We will simply hit this problem with a hammer and take the total complex of a Cartan-Eilenberg resolution.

Define \( R^n F(C') = H^n F(I') \). There is a hidden exercise here, which is to check that this is independent of the choice of \( I \).

Proposition 17. If \( A \to B \) is a quasi-isomorphism of bounded-below cochain complexes and \( \alpha : A \to I \) such that \( I \in \text{Ch}(\text{inj}(\mathcal{A})) \) then \( \exists \beta : B \to I \) and a homotopy equivalence \( h : \beta \circ \varphi = \alpha \), i.e. the following diagram commutes up to homotopy.
Proof.

\[ \begin{array}{c}
A \\
\downarrow \phi \\
B
\end{array} \xrightarrow{\alpha} \begin{array}{c}
I \\
\downarrow \beta \\
J
\end{array} \]

\[ \Phi \circ \Psi \approx \Psi \circ \Phi \]

Corollary 5. If \( I \xrightarrow{\varphi} J \) is a quasi-isomorphism of bounded below cochain complexes, then it has an inverse up to homotopy.

Proof. Consider the diagram below.

\[ \begin{array}{c}
I \\
\downarrow \phi \\
J
\end{array} \xrightarrow{id} \begin{array}{c}
I \\
\downarrow \psi \\
J
\end{array} \]

So we get that \( \psi \varphi = 1_I \). Similarly, we can find \( \psi' \) such that \( \varphi \psi' = 1_J \). Then \( \psi \approx \psi' \circ \varphi \circ \psi' \). So we get a homotopy inverse \( \psi \).

Definition 29.

\[ D^+(A) = \frac{\text{Ch}^+ (\text{inj}(A))}{\sim} \]

Similarly, we can define

\[ D^-(A) = \frac{\text{Ch}^- (\text{proj}(A))}{\sim}, \]

for cochain complexes which vanish above zero and projective objects in \( A \), assuming \( A \) has enough projectives. More generally, we have

\[ D^b(A) = \frac{\text{Ch}^b (\text{inj}(A))}{\sim} = \frac{\text{Ch}^b (\text{proj}(A))}{\sim}, \]

where the \( b \) denotes complexes with bounded cohomology.

Lecture 20: Existence of Derived Categories - 03/20/2020

Last time we alluded to the universal property of derived categories. To precisely define this property will take some work. The reason for this is that the universal property satisfied in this case is '2-categorical'.

Definition 30. Let \( A \) be an abelian category. The derived category of \( A \) is the universal category \( D(A) \) and functor \( \Phi : \text{Ch}(A) \rightarrow D(A) \) such that \( \Phi(\text{quasi-iso}) \subset \text{iso} \). That is if \( (\mathcal{C}, \Psi) \) is another category-functor pair such that \( \Psi(\text{quasi-iso}) \subset \text{iso} \), then there exists a functor \( \Gamma \) and a natural isomorphism \( \gamma : \Gamma \circ \Phi \approx \Psi \). Diagrammatically, we have:

\[ \begin{array}{c}
\text{Ch}(A) \\
\downarrow \Phi \\
D(A)
\end{array} \xrightarrow{\exists \gamma} \begin{array}{c}
\mathcal{C} \\
\downarrow \Psi \\
\mathcal{D}
\end{array} \xrightarrow{\exists \Gamma} \]

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The pair \((\Gamma, \gamma)\) is unique up to unique isomorphism, that is, if \((\Gamma', \gamma')\) is another such datum then there is a unique natural isomorphism of functors \(h: \Gamma \cong \Gamma'\) such that the following diagram commutes.

\[
\begin{array}{ccc}
\text{Ch}(\mathcal{A}) & \xrightarrow{\Phi} & \mathcal{D}(\mathcal{A}) \\
\Psi \downarrow & & \downarrow \exists h \\
\mathcal{C} & \xrightarrow{\Gamma} & \Gamma'
\end{array}
\]

**Proposition 18.** If \((\mathcal{D}, \Phi)\) satisfies this universal property then \((\mathcal{D}, \Phi)\) is unique up to an equivalence that is unique up to unique isomorphism. Explicitly, if \((\mathcal{D}', \Phi')\) is another category-functor pair satisfying the universal property of the derived category then there is an equivalence of categories \(\Gamma: \mathcal{D} \rightarrow \mathcal{D}'\) such that \(\Gamma \circ \Phi = \Phi'\). Furthermore, if \(\Gamma'\) is another such equivalence, then there is a unique natural isomorphism \(\eta: \Gamma \rightarrow \Gamma'\) making the diagram below commute.

\[
\begin{array}{ccc}
\text{Ch}(\mathcal{A}) & \xrightarrow{\Phi} & \mathcal{D}(\mathcal{A}) \\
\Phi' \downarrow & & \downarrow \exists \eta \\
\mathcal{D} & \xrightarrow{\Gamma} & \mathcal{D}'
\end{array}
\]

The proof of this is left to the reader. We might come back to it later, but for now we will focus on constructing the derived category of a large class of abelian categories. We note that in general the derived category of any abelian category need not exist.

**Definition 31.** An abelian category \(\mathcal{A}\) has a **generator** if there is an object \(X\) of \(\mathcal{A}\) such that every object of \(\mathcal{A}\) is a quotient of a direct sum of copies of \(X\).

In general, most everyday categories are going to have a generator. It would take a fairly large and pathological category to not satisfy this.

**Theorem 12.** If \(\mathcal{A}\) is an abelian category satisfying:

1. \(\mathcal{A}\) has all direct sums (is complete)
2. filtered colimits are exact
3. \(\mathcal{A}\) has a generator

Then \(\mathcal{D}(\mathcal{A})\) exists.

**Remark 6.** These hypotheses also guarantee that \(\mathcal{A}\) has enough injectives (Grothendieck). The proof is very interesting and the reader is encouraged to read up on it.

**Proof.** Let \(\mathcal{D} = \mathcal{D}(\mathcal{A})\). Define

\[\text{Ob}(\mathcal{D}) = \text{Ob}(K(\mathcal{A})) = \text{Ob}(\text{Ch}(\mathcal{A})),\]

and

\[\text{Hom}_{\mathcal{D}}(X, Y) \lim_{\xrightarrow{X' \rightarrow X}} \text{Hom}_{K(\mathcal{A})}(X', Y),\]

here the direct limit is over quasi-isomorphisms \(X' \rightarrow X\). There are a couple of things to check here. For one, how do we compose these morphisms? Another thing to check is that the collection of \(X'\) quasi-isomorphic to \(X\) is small, or that there is an equivalent colimit indexed over a small set. We will address this latter concern first using the following lemma.
**Lemma 9.** The diagram of quasi-isomorphisms $X' \xrightarrow{\sim} X$ is essentially small.

**Proof.** We want to find $X'' \xrightarrow{\sim} X'$ such that $X''$ has bounded cardinality for any such $X'$. (Note: in the category of $R$-modules, this means precisely what it sounds like, in other categories one would need to make sense of this, but we will ignore this for now.) Look at all subcomplexes $X'_\mu \subset X'$ and note that

$$X' = \lim_{\mu} X'_\mu.$$  

Then since filtered colimits in $\mathcal{A}$ are exact,

$$H^*(X') = \lim_{\mu} H^*(X'_\mu).$$

Now choose $\lambda \in \mathbb{N}$ such that every $X'$ is a quotient of $A_0^{\oplus \lambda}$, where $A_0$ is a generator for $\mathcal{A}$. Missed some stuff here Then $X'_\mu \xrightarrow{\sim} X$ and $|X'_\mu|$ is bounded in terms of $|H^*(X)|$. So complexes of bounded size are essentially small.

That composition is defined will follow from the following lemma.

**Lemma 10.** Given the solid arrow diagram below, where $t$ is a quasi-isomorphism, there exists an object $X''$ and the dashed morphisms where $t'$ is a quasi-isomorphism.

(Include the diagram here)

**Lemma 10.** Given the solid arrow diagram below, where $t$ is a quasi-isomorphism, there exists an object $X''$ and the dashed morphisms where $t'$ is a quasi-isomorphism.

**Todo:** Complete proof

---

**Lecture 21: Derived Categories (continued) 03/30/2020**

First day back from spring break!

Recall our construction of the derived category of an abelian category. If $\mathcal{A}$ is an abelian category, we built $\mathcal{D}(\mathcal{A})$ by defining

$$\text{Ob}(\mathcal{D}(\mathcal{A})) = \text{Ob}(\mathcal{K}(\mathcal{A})) = \text{Ob}(\text{Ch}(\mathcal{A})).$$

Further,

$$\text{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) = \lim_{X \text{ quasi-iso}} \text{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y).$$

Recall that $\text{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y)$ is the chain homotopy classes of chain maps $X \rightarrow Y$. We proved that the colimit above makes sense and that composition is well-defined so that we do indeed get a category. We also saw that the derived category satisfies a universal property. Namely, if $\text{Ch}(\mathcal{A}) \xrightarrow{\Psi} \mathcal{C}$ and $\Psi$ takes quasi-isomorphisms to isomorphisms. Then there exists a factorization $\Gamma : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{C}$ of this functor through the derived category and that this factorization $\Gamma$ is unique up to unique natural isomorphism.

**Proposition 19.** $\mathcal{D}(\mathcal{A})$ satisfies this universal property.
Proof. Suppose \( \text{Ch}(\mathcal{A}) \xrightarrow{\Psi} \mathcal{C} \) and \( \Psi \) takes quasi-isomorphisms to isomorphisms. Let \( \Gamma(X) = \Psi(X) \).
That was easy, the real work is in defining the functor on morphisms. We need a map
\[
\text{Hom}_{\text{Ch}(\mathcal{A})}(X, Y) \to \text{Hom}_{\mathcal{C}}(X, Y).
\]
This will act by
\[
(X \to X' \to Y) \mapsto \Psi(f) \circ \Psi(s)^{-1},
\]
where \( s \) is the quasi-isomorphism going to the left and \( f \) is the morphism on the right. We need to check a couple of things. For one, we need to check that this is well-defined on chain homotopy equivalence classes. Additionally, one would have to check that this formula is independent of the choice of representatives \((s, f)\). Lastly, one needs to check that \((\Gamma, \gamma)\) is unique up to unique isomorphism.

\[\square\]

**Corollary 6.** In \( \mathcal{D}(\mathcal{A}) \),
\[
\text{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) = \lim_{X' \text{ quasi } X} \text{Hom}_{K(\mathcal{A})}(X, Y).
\]

(TODO: isn’t this a definition?)
Recall from one of the homework that \( K(\mathcal{A}) \) is not an abelian category. This raises the question of what the structure of \( \mathcal{D}(\mathcal{A}) \) looks like. Is the derived category an abelian category? It turns out that the functor \( \text{Ch}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) \) preserves finite direct sums. From this, it follows that \( \mathcal{D}(\mathcal{A}) \) has finite products, finite coproducts, and that these coincide. To see this, note that
\[
\text{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y \times Z) = \lim_{X' \text{ quasi } X} \text{Hom}_{K(\mathcal{A})}(X', Y \times Z) = \lim_{X' \text{ quasi } X} \left( \text{Hom}_{K(\mathcal{A})}(X', Y) \times \text{Hom}_{K(\mathcal{A})}(X', Z) \right).
\]
This last step above is where we use the fact that the functor \( \text{Ch}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) \) preserves products. Then because sifted colimits commute with products,
\[
\lim_{X' \text{ quasi } X} \text{Hom}_{K(\mathcal{A})}(X', Y) \times \lim_{X' \text{ quasi } X} \text{Hom}_{K(\mathcal{A})}(X', Z)
\]
\[
= \text{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) \times \text{Hom}_{\mathcal{D}(\mathcal{A})}(X, Z).
\]
The same argument shows that 0 is an initial object in \( \mathcal{D}(\mathcal{A}) \). To see this, simply replace \( Y, Z \) with 0 in the argument. A symmetric argument yields that coproducts are preserved, from which it follows that these coincide in the derived category. Similarly, this symmetric argument modified shows that 0 is also a final object in \( \mathcal{D}(\mathcal{A}) \). Thus, modulo a few details, we have the following result.

**Corollary 7.** \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) \) has a commutative monoid structure with unit.

We actually have more than this. Recall how we constructed inverses in \( \text{Hom}_{\mathcal{A}}(X, Y) \). This is displayed visually below.
In $\mathcal{A}$, as in $\text{Ch}(\mathcal{A})$, we have a split exact sequence

$$0 \to X \to X \oplus X \to X \to 0.$$ 

Since, as we just proved, $\text{Ch}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ preserves direct sums, this sequence is split exact in $\mathcal{D}(\mathcal{A})$ as well. What do we mean by this exactly? We then get $-1 : X \to X$ in $\mathcal{D}(\mathcal{A})$, and

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 0$$

implies that $-1 + 1 = 0$. Using this, we get $-f = -1 \circ f$.

**Corollary 8.** $\text{Hom}_{\mathcal{D}(\mathcal{A})}(X,Y)$ is an abelian group so that $\mathcal{D}(\mathcal{A})$ is an additive category.

But what about kernels and cokernels?

**Example 16.** Let’s compute $\text{coker}(\mathbb{Z} \to \mathbb{Z})$ in $\mathcal{D}(\mathcal{A})$ (or $\mathcal{K}(\mathcal{A})$). Let’s pretend that this exists, so that we can write $Q = \text{coker}(\mathbb{Z}[0] \to \mathbb{Z}[0])$.

We then have

$$\text{Hom}_{\mathcal{K}(\mathcal{A})}(Q, P) = \{ \alpha : \mathbb{Z}[0] \to P | \alpha \circ n = 0 = | \alpha \in P_0 | d(\alpha) = 0, \exists \beta \in P_1, d(\beta) = n \alpha \}.$$

**Exercise 48.** Compute $H_*(Q_*)$ using Pontryagin duality:

$$\text{Hom}(H_*(Q_*), \mathbb{Q}/\mathbb{Z}) = H^* \text{Hom}(Q_*, \mathbb{Q}/\mathbb{Z}).$$

**Lecture 22: Sifted Categories - 04/01/2020**

Today we will go back and fill in some details from the last lecture. Last time we mentioned sifted colimits. Let us go through what these are exactly.

**Definition 32.** A category $I$ is called cosifted if for all $i, j \in \text{Ob}(I)$, the category of all diagrams of the form

$$i \quad \xrightarrow{k} \quad j$$

is connected. Such a diagram is called a cospan. A morphism of cospans is a commutative diagram like the one below.

$$i \quad \xleftarrow{k'} \quad k \quad \xrightarrow{k} \quad j$$
I is called **sifted** if the same statement holds with arrows reversed. The diagrams in this case are called **spans**.

**Theorem 13.** If \( I \) is cosifted, then and \( X, Y : I \to \text{Set} \). Then

\[
\lim_{\leftarrow} (X_i \times Y_i) \to \lim_{\leftarrow} (X_i) \times \lim_{\leftarrow} (Y_i)
\]

is a bijection.

**Proof:** Recall that \( \lim_{\leftarrow} (X_i) \) can be constructed as a quotient of \( \coprod X_i \) by the relation \( (i, \alpha) \sim (j, \beta) \) if there is a sequence of morphisms in \( I \) taking \( i \) to \( j \). The image under \( X \) of these morphisms then gives a sequence of morphisms in \( \text{Set} \) taking \( \alpha \in X_i \) to \( \beta \in X_j \).

We construct an inverse. For \( (k, (u_*(\alpha), v_*(\beta))) \in X_k \times Y_k \), choose a cospan.

\[
((i, \alpha), (j, \beta)) \to (k, (u_*(\alpha), v_*(\beta))
\]

Now let’s go back to the derived category and apply this idea.

**Theorem 14.** In \( K(\mathcal{A}) \), the category of quasi-isomorphisms \( X' \to X \) is sifted.

**Proof:** We have to show that in the following diagram with dark arrows in which all arrows are isomorphisms, the blue arrows and \( V \) exist.

\[
\begin{array}{ccc}
W & \xrightarrow{f} & W' \\
\downarrow{g} & \searrow{f'} & \downarrow{g'} \\
Y & \xleftarrow{p} & Z \\
\downarrow{q} & & \downarrow{q'} \\
X & \xleftarrow{h} & & \xrightarrow{h'}
\end{array}
\]

\[
h : qg \sim pf, \quad h' : qg' \sim pf'.
\]

The idea is to define

\[
V = \{w \in W, w' \in W', s : f(w) \sim f'(w'), t : g(w) \sim g'(w'), H : ps \simeq qt\}.
\]

Concretely, we let

\[
V_n = W_n \oplus W'_n \oplus Y_{n+1} \oplus Z_{n+1} \oplus X_{n+2}
\]

with differential

\[
d = \begin{pmatrix}
d & 0 & -d & f & -f' & d & g & g' & 0 & d & \pm h & \pm h' & p & -q & d
\end{pmatrix}.
\]
commutes up to homotopy. So we construct an explicit chain homotopy \( S : V_n \rightarrow Y_{n+1} \) by just taking it to be the projection of \( V_n \) onto the appropriate factor. One easily verifies that \( dS \pm Sd = f' - f \).

**Lecture 23: Triangulated Categories - 04/03/2020**

In the derived category, kernels and cokernels are replaced by fibers and cones. These are related by fiber = cone \([-1]\). Note that if \( X \rightarrow Y \) is injective, we can recover \( X = \ker(Y \rightarrow \text{coker}) \). In general we can’t do this. In the cokernel, we identify the image of any element of \( X \) with 0, but we forget which element of \( X \) gave the identification. *(TODO: Fill in some details that were missed due to technical difficulties)*.

**Definition 33.** A **triangulated category** is an additive category, \( \mathcal{D} \), with an automorphism, \( T : \mathcal{D} \rightarrow \mathcal{D} \) (i.e. a functor to itself that gives an equivalence). The action of this functor is usually denoted by \( T(X) = X[1] \), and called the shift. There is also a collection of triples \((u, v, w)\) usually denoted \( X \rightarrow Y \rightarrow Z \rightarrow X[1] \) and called exact triangles. Here \( w : Z \rightarrow X[1] \). Additionally, this data is subject to the following axioms:

1. (a) Every \( u : X \rightarrow Y \) extends to an exact triangle.
   (b) \( X \xrightarrow{id_X} X \rightarrow X[1] \) is an exact triangle
   (c) any triangle is isomorphic to an exact triangle.
2. If \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \) is exact, then so are
   \[ Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u} Y[1] \]
   and
   \[ Z[-1] \xrightarrow{-w} X \xrightarrow{u} Y \xrightarrow{v} Z. \]
3. Given a solid arrow commutative diagram below, where both horizontal rows are exact triangles, the dashed arrow exists and preserves commutativity of the diagram. Note: this morphism is not necessarily unique.

\[
\begin{array}{c}
X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \\
\downarrow{f} \downarrow{g} \downarrow{1} \downarrow{f(1)} \\
X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{1} X'[1]
\end{array}
\]

4. The octahedral axiom: Given three exact triangles
   \[ X \xrightarrow{f} Y \xrightarrow{u} Z \xrightarrow{d} X[1], \]
There exists a fourth exact triangle

\[ Z' \xrightarrow{\phi} Y' \xrightarrow{\psi} X' \xrightarrow{\theta} Z'[1] \]

such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{gf} & Z \\
\downarrow{f} & & \downarrow{g} \\
Y & & Z' \\
\downarrow{u} & & \downarrow{d''} \\
Z' & \xrightarrow{d} & X[1]
\end{array}
\]

\[
\begin{array}{ccc}
X' & \xrightarrow{\theta} & Z'[1] \\
\downarrow{d'} & & \downarrow{u} \\
X' & \xrightarrow{\psi} & Y[1] \\
\downarrow{\phi} & & \downarrow{\phi} \\
Y' & \xrightarrow{\psi} & X[1]
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{u} & Z \\
\downarrow{f} & & \downarrow{f} \\
Z' & \xrightarrow{u} & Y' \\
\downarrow{f} & & \downarrow{f} \\
X'[1] & \xrightarrow{u} & X[1]
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{u} & Z \\
\downarrow{f} & & \downarrow{f} \\
Z' & \xrightarrow{u} & Y' \\
\downarrow{f} & & \downarrow{f} \\
X'[1] & \xrightarrow{u} & X[1]
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{u} & Z \\
\downarrow{f} & & \downarrow{f} \\
Z' & \xrightarrow{u} & Y' \\
\downarrow{f} & & \downarrow{f} \\
X'[1] & \xrightarrow{u} & X[1]
\end{array}
\]

**Remark 7.** In an abelian category, the dashed arrows below exist by the universal property of the cokernel. This suggests that the octahedral axiom is the snake lemma in a triangulated category.

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{u} & Z \\
\downarrow{f} & & \downarrow{f} \\
Z' & \xrightarrow{u} & Y' \\
\downarrow{f} & & \downarrow{f} \\
X'[1] & \xrightarrow{u} & X[1]
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{u} & Z \\
\downarrow{f} & & \downarrow{f} \\
Z' & \xrightarrow{u} & Y' \\
\downarrow{f} & & \downarrow{f} \\
X'[1] & \xrightarrow{u} & X[1]
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{u} & Z \\
\downarrow{f} & & \downarrow{f} \\
Z' & \xrightarrow{u} & Y' \\
\downarrow{f} & & \downarrow{f} \\
X'[1] & \xrightarrow{u} & X[1]
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{u} & Z \\
\downarrow{f} & & \downarrow{f} \\
Z' & \xrightarrow{u} & Y' \\
\downarrow{f} & & \downarrow{f} \\
X'[1] & \xrightarrow{u} & X[1]
\end{array}
\]

**Theorem 15.** If \( \mathcal{A} \) is abelian, \( K(\mathcal{A}) \) and \( D(\mathcal{A}) \) are triangulated where \( X \xrightarrow{u} Y \rightarrow Z \rightarrow X[1] \) exact if it is isomorphic to \( X \xrightarrow{u} Y \rightarrow \text{cone}(u) \rightarrow X[1] \).

**Proof.** Axiom 1(a) follows from the fact that any map has a cone. Axiom 1(b) follows from the fact that \( \text{cone}(id_X) \simeq 0 \) for any object \( X \). Axiom 1(c) follows by our definition. Axiom 2 follows from the fact that for any morphism \( u : X \rightarrow Y \),

\[
\text{cone}(Y \rightarrow \text{cone}(u)) \simeq X.
\]

Axiom 3 follows from the fact that the cone construction is functorial.

**Exercise 49.** Weibel 10.1.2 Solution

**Exercise 50.** Weibel 10.2.1 Solution

**Exercise 51.** Weibel 10.2.2 Solution

**Exercise 52.** Weibel 10.4.2 Solution
Now that we have the derived category, today we will spend some time seeing how it fits into the framework of some of the things we’ve done so far.

**Definition 34.** A **triangulated functor** between triangulated categories is a functor \( F : \mathcal{D} \to \mathcal{D}' \) such that

1. \( F(\text{exact triangles}) \subseteq \text{exact triangles} \)
2. \( F(X[1]) = F(x)[1] \)

**Exercise 53.** If \( F \) is a triangulated functor \( \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \) then \( \{ H_n(F(q(X))) \}_{n \in \mathbb{Z}} \) is a \( \delta \)-functor, where \( q : \mathcal{A} \to \mathcal{D}(\mathcal{A}) \) is the natural functor.

Our goal is to start with \( F : \mathcal{A} \to \mathcal{B} \) and extend this to a triangulated functor \( RF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \). The notation here is meant to be suggestive and should remind you of derived functors!

First note that \( F \) extends automatically to \( \text{Ch}(\mathcal{A}) \) by just applying \( F \) to each object and morphism which builds up a chain complex. Further, \( F \) (a chain homotopy) is also a chain homotopy (Note: here, as in most of these lectures, we are assuming \( F \) is additive). Thus, we automatically have an extension \( F : K(\mathcal{A}) \to K(\mathcal{B}) \) which we will also denote by \( F \). Now the question is, does the dashed arrow below exist?

\[
\begin{array}{ccc}
K(\mathcal{A}) & \xrightarrow{F} & K(\mathcal{B}) \\
\downarrow q & & \downarrow q \\
\mathcal{D}(\mathcal{A}) & \xrightarrow{RF} & \mathcal{D}(\mathcal{B})
\end{array}
\]

The answer to this question is: of course not!

**Example 17.** Let \( \mathcal{A} \) be the category of abelian groups and let \( F(X) = \mathbb{Z}/n \mathbb{Z} \otimes X \). Then \( F(\mathbb{Z}/n \mathbb{Z}[0]) = \mathbb{Z}/n \mathbb{Z}[0] \), but \( \mathbb{Z}/n \mathbb{Z} \cong [\mathbb{Z} \xrightarrow{n} \mathbb{Z}] \) and

\[
F([\mathbb{Z} \xrightarrow{n} \mathbb{Z}]) = [\mathbb{Z}/n \mathbb{Z} \xrightarrow{0} \mathbb{Z}/n \mathbb{Z}].
\]

So this extension idea was a bust, but maybe we can try to approximate an extension.

**Definition 35.** A **right derived functor** of a triangulated functor

\[
F : K(\mathcal{A}) \to K(\mathcal{B}),
\]

is a functor

\[
RF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})
\]

and a natural transformation

\[
\eta : qF \to RF \circ q
\]

such that the pair \((RF, \eta)\) is universal. A **left derived functor** of a triangulated functor \( F \) as above is a functor \( LF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \) such with a natural transformation \( \eta' : RF \circ q \to LF \).

**Example 18.** Going back to the example above... **TODO: missed this.**

Now let us see how to actually construct these derived functors in this general setting. The idea is to find a triangulated subcategory \( K' \) of \( K(\mathcal{A}) = : K \) such that
1. \( \forall x \in K, \exists \) a quasi-isomorphism \( X \to X' \in K' \)

2. If \( X' \in K' \) and \( X' \) is quasi-isomorphic to 0, then \( F(X') \simeq 0 \).

**Remark 8.** If \( X' \xrightarrow{f} X'' \) is a quasi-isomorphism, then complete it to an exact triangle
\[
X' \to X'' \to \text{cone}(f) \to X'[1].
\]

Note \( \text{cone}(f) \simeq 0 \). Then applying \( F \) to this sequence gives us that \( F(X') \to F(X'') \) is also a quasi-isomorphism.

**Definition 36.** Let \( RF(X) = F(X') \) where \( X \to X' \) is as in 1. above.

We need to show that this definition is independent of our choice of \( X' \) and quasi-isomorphism. **TODO:** this.

**Theorem 16.** Suppose there exists a triangulated \( K' \subset K \) such that 1. and 2. above hold. Then
\[
RF : \mathcal{D} \to \mathcal{D}' \to \mathcal{D}(\mathcal{B})
\]
is a right derived functor of \( F \).

**Remark 9.** Here we are implicitly using the fact that there is actually an extension \( F : \mathcal{D}' \to \mathcal{D}(\mathcal{B}) \). The idea is that we are moving down to this smaller category \( K' \) where the extension problem actually works out well and then use this extension to obtain our 'approximation'.

Now lets pivot back to the topic of spectral sequences. Let \( C' \) be a chain complex in \( \mathcal{A} \). The process
\[
C' \leadsto q(C') \in \mathcal{D}(\mathcal{A})
\]
looses some information. To see this, first note that \( C' \) has two filtrations
\[
\cdots \subset \sigma^{\geq n+1}C' \subset \sigma^{\geq n}C' \subset \cdots \subset C'
\]
and
\[
\tau_{\leq n}C' = [\cdots \to C^{n-1} \to Z^n C' \to 0 \to \cdots] \subset \tau_{\leq n+1}C'.
\]

For historic reasons, we mention here that \( \tau \) stands for 'truncation' and \( \sigma \) stands for 'stupid truncation'. The reason \( \sigma \) gets an insulting name is because it does not behave well with respect to homology, i.e. it changes the homology in each degree.

Suppose now that \( \Gamma : \mathcal{A} \to \mathcal{B} \) is left exact. Then \( \Gamma(F,I') \) gives a filtration of \( \Gamma(I') = RF(C') \) so \( \Gamma(F,I') \) has a spectral sequence
\[
H^p(\text{gr}_q \Gamma(I')) \Rightarrow R^p \Gamma(C').
\]

We will continue with this idea next time.
This is the last lecture we will spend on the topic of derived categories. We will use this to say some last minute things that were forgotten in the previous lectures. One thing to note is that not every triangulated category is the derived category for some abelian category. A prime example of this is the stable homotopy category. This implies that for a general triangulated category \( \mathcal{D} \) there is not a canonical homology functor \( H_n : \mathcal{D} \to \mathcal{A} \). However, we do have the functors \( \text{Hom}(\_ , X) \) for each object \( X \) of \( \mathcal{D} \).

**Proposition 20.** Suppose that
\[
X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]
\]
is an exact triangle in a triangulated category \( \mathcal{D} \). Then for all objects \( W \) of \( \mathcal{D} \), if we apply the functor \( \text{Hom}(W, \_ ) \), we get an exact sequence
\[
\text{Hom}(W, X) \to \text{Hom}(W, Y) \to \text{Hom}(W, Z).
\]

**Proof.** Suppose \( \alpha \in \ker(\text{Hom}(W, Y) \to \text{Hom}(W, Z)) \). Then we have a diagram like the following in which the dashed arrow exists by virtue of axiom TR3 in Weibel.

\[
\begin{array}{ccc}
0 & \to & W \\
\downarrow & & \downarrow \text{ } 1 \\
Z[-1] & \to & X \\
\downarrow \beta & & \downarrow & \to Y \to Z
\end{array}
\]

**TODO: Finish this proof.**

**Claim:** \( \text{Hom}(W, X[n]) = \text{Ext}^n(W, X) \). How do we see this? Well we know that the functor
\[
\text{Hom} : \mathcal{A}^{op} \times \mathcal{A} \to \text{Ab}
\]
is left exact in each variable. It extends to
\[
\text{Hom} : K(\mathcal{A})^{op} \times K(\mathcal{A}) \to \text{Ab}
\]
and even
\[
\text{Hom} : \mathcal{D}(\mathcal{A})^{op} \times \mathcal{D}(\mathcal{A}) \to \text{Ab}.
\]
But we also have an enriched hom:
\[
\text{Hom}^* : \text{Ch}(\mathcal{A})^{op} \times \text{Ch}(\mathcal{A}) \to \text{Ch}(\text{Ab})
\]
where
\[
\text{Hom}^n(X', Y') = \prod_{m \in \mathbb{Z}} \text{Hom}(X'^m, Y'^{n+m}).
\]
This functor plays nicely with chain homotopies and induces an enriched triangulated hom
\[
\text{Hom}^* : K(\mathcal{A})^{op} \times K(\mathcal{A}) \to K(\text{Ab}).
\]
We can then try to derive this functor. Assuming that \( \mathcal{A} \) has enough projectives, we get the following functor
\[
R \text{Hom}(\_ , Y^* ) : \mathcal{D}^{-}(\mathcal{A})^{op} \to \mathcal{D}(\mathcal{A}),
\]

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and assuming that \( \mathcal{A} \) has enough injectives, we also have

\[
R \text{Hom}(X, \_): \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}(\text{Ab}).
\]

We expect to find a diagram like the following which gives us a derived functor given by the dashed arrow. (TODO: draw diagram). We want to show that

\[
R \text{Hom}(\_, Y)(X')
\]

factors as a functor of \( Y^\cdot \in K^+(\mathcal{A}) \), through \( D^+(\mathcal{A}) \). (TODO: fill in details missed here. The point is to show that Homs in the derived category correspond to ext groups.)

We will say one last thing about derived categories in relation to spectral sequences. It appears that there is not a great way to obtain a spectral sequence from the derived category. Indeed, this is one of the advantages of the \( \infty \)-categories. Let \( C^\cdot \in \text{Ch}^+(\mathcal{A}) \). This object has canonical filtrations, which we have mentioned before as being given by \( \tau \) or \( \sigma \) truncations. These correspond to maps

\[
\ldots \leftarrow \sigma^{\geq n}C^\cdot \leftarrow \sigma^{\geq n+1}C^\cdot \leftarrow \ldots
\]

in \( \mathcal{D}(A) \). (TODO: fill in triangles above.) Then if \( \Gamma : \mathcal{D}(A) \to \mathcal{D}(B) \) is a triangulated functor then \( R\Gamma \) will preserve these triangles so that \( R\Gamma(C^\cdot) \) as a filtered object of \( \mathcal{D}(B) \).

**Theorem 17.** Assume you have a sequence of functors \( \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \) along with adapted classes \( K'(\mathcal{A}) \subset K(\mathcal{A}), K'(\mathcal{B}) \subset K(\mathcal{B}) \) and \( F(K'(\mathcal{A})) \subset K'(\mathcal{B}) \), then we get

\[
RG \circ RF = R(G \circ F).
\]

---

**Lecture 26: Group Cohomology - 04/13/2020**

**Lecture 27: Group Cohomology (continued) - 04/15/2020**

Today we will continue computing the group cohomology in particular cases of interest. The first result of these is the following. As usual, we work within an abelian category \( \mathcal{A} \) with enough projectives and injectives to define derived functors.

**Proposition 21.** If \(|G| \) acts invertibly on \( M \), then \( H^n(G,M) = H_n(G,M) = 0 \) and \( H^0(G,M) = H_0(G,M) = N(m) \).

**Proof.** Let \( G - \mathcal{A} \) be the category of \( G \)-objects of \( \mathcal{A} \), and \( G - \mathcal{A}[m^{-1}] \) the full subcategory where \( m = |G| \) acts invertibly. Our strategy will be to compare the group cohomology for these two cases and use the latter to compute the former. We start with the forgetfull functors

\[
F : \mathcal{A}[m^{-1}] \to \mathcal{A}, \quad \text{and} \quad F : G - \mathcal{A}[m^{-1}] \to G - \mathcal{A}.
\]

The first of these is just the special case that \( G = \{1\} \). These admit a right adjoint

\[
\Phi(P) = \lim_k m^k P
\]

and a left adjoint

\[
\Psi(P) = \lim_k m^{-k} P.
\]

It follows that \( F \) is exact. Now \( \Phi \), being a right adjoint, preserves injectives. Let \( \Gamma : G - \mathcal{A} \to \mathcal{A} \) be \( \Gamma(M) = M^G \). This is now precisely the data we need to get a Grothendieck spectral sequence.

\[\square\]
Recall that we have the norm element $N = \sum_{g \in G} g$ in $\mathbb{Z}G$. It is not too hard to see that $N^2 = \sum_{g, h \in G} gh = mN$, where $m = |G|$. It follows that $e = m^{-1}N$ is an idempotent so that there is a direct sum decomposition for any module $M = eM \oplus (1 - e)M$.

**Lemma 11.** If $|G|$ acts invertibly on $M$, then $eM = M^G = M_G$.

**Proof.** TODO: the proof.

So we find that in $G - \mathcal{A}[m^{-1}]$, every object splits functorially into

$$M = M^G \oplus (1 - e)M = M_G \oplus (1 - e)M.$$ 

It follows that $(\_)^G$ and $(\_)_G$ are exact functors. It follows that the group cohomology in this case vanishes above degree zero.

Now let’s move on to what the higher cohomology groups actually tell us. Let’s start with cohomology in degree one. We know that $H^1(G, M) = \text{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, M)$. So the elements of this cohomology group are extensions:

$$0 \to M \xrightarrow{\varphi} X \xrightarrow{\psi} \mathbb{Z} \to 0.$$ 

Choose $x \in X$ such that $\varphi(x) = 1$. Now let $\varphi : G \to M$ be $\varphi(g) = g \cdot x - x$. This is not a homomorphism, so why do we care? Well, we have

$$\varphi(g h) = gh \cdot x - x = g \cdot (hx - x) + (ax - x) = g \cdot \varphi(h) + \varphi(g).$$ 

We see that this is almost a homomorphism, it is in fact a homomorphism up to this action of $g$. We call such a map a **crossed homomorphism**. Let $\text{Der}(G, M)$ be the set of crossed homomorphisms $G \to M$. It appears now that we have a map $H^1(G, M) \to \text{Der}(G, M)$. This turns out to almost give us an isomorphism. To see this, first we ask what happens if we replace this $x$ by $x'$? Write $x' = x + y$ for $y \in M$. Let $\varphi'(x) = g \cdot x' - x'$ and $\psi(g) = g \cdot y - y$. Then a quick computation shows that

$$\varphi'(g) = \varphi(g) + \psi(g).$$ 

This shows the following.

**Theorem 18.** There is an isomorphism

$$H^1(G, M) \xrightarrow{\sim} \text{Der}(G, M)/M.$$ 

**Proof.** The inverse is given by $X = M \times \mathbb{Z}$ with action given by

$$g \cdot (x, n) = (n \varphi(g) + g \cdot x, n).$$

What about homology? Suppose that $I$ is an injective abelian group. Then by the universal coefficient theorem,

$$\text{Hom}(H_1(G, \mathbb{Z}), I) = H^1(G, I) = \text{Der}(G, I)/I.$$ 

But the $G$-action on $I$ is trivial so (TODO: finish the argument here).

**Lemma 12.** If $\text{Hom}(X, \_)$ and $\text{Hom}(Y, \_)$ are naturally isomorphic functors on the subcategory of injective objects of $\mathcal{A}$, and $\mathcal{A}$ has enough injectives, then $X \equiv Y$.

TODO: Fix up this proof.
First, we go back and fix the result from Proposition 21.

**Proposition 22.** Suppose \( G \) is a group, \( m \in \mathbb{Z} \), \( m \) acts invertibly on a \( G \)-module \( A \). Then

\[
H^*(G,A) = H^*(G[m^{-1}],A)
\]

and

\[
H_*(G,A) = H_*(G[m^{-1}],A).
\]

**TODO: what is going on with this proof?** Now let's turn to the meaning of \( H^*(G,A) \). Let \( I = \ker(\mathbb{Z}[G] \to \mathbb{Z}) \) be the augmentation ideal which is generated by elements of the form \( g-1 \) for \( g \in G \). From the long exact sequence associated to

\[
0 \to I \to \mathbb{Z}G \to \mathbb{Z} \to 0
\]

we get that

\[
H^n(G,A) = \text{Ext}^n_{G\text{-mod}}(\mathbb{Z},A) = \text{Ext}^{n-1}(I,A).
\]

Further, we get

\[
H^1(G,A) = \frac{\text{Hom}(I,A)}{\text{Hom}(\mathbb{Z}[G],A)} = \frac{\text{Hom}(I,A)}{A}.
\]

**Proposition 23.**

\[
\text{Hom}_G(I,A) = \text{Der}(G,A)
\]

**Proof.** The idea is that the map \( G \xrightarrow{\psi} I \), given by \( g \mapsto g-1 \) is the universal crossed homomorphism. It is in fact a crossed homomorphism, as one readily checks. Then for any homomorphism \( I \to A \), \( \psi \circ \phi \) is a crossed homomorphism. For the inverse map, suppose \( a : G \to A \) is a crossed homomorphism. Let \( \psi(g-1) = a(g) \). One can check that this is in fact a homomorphism of \( G \)-modules and gives an inverse.

Note that we obtain Theorem 18 as a corollary of this. What about \( \text{Ext}^1(I,A) \)? Let's define \( \text{CrExt}^1(G,A) \) to be the set of extensions

\[
0 \to A \to X \to G \to 0
\]

such that the \( G \)-action on \( A \) is just the conjugation action. We call such extensions crossed extensions and claim that \( H^2(G,A) = \text{CrExt}^1(G,A) \). Start with an extension

\[
0 \to A \to X \xrightarrow{p} I \to 0
\]

and the universal map \( G \to I \) given by \( g \mapsto g-1 \). Then taking the pullback of this map along \( p \) gives \( Y = \{(x,g) | p(x) = g-1\} \) with a canonical projection \( q : Y \to G \) sending \((x,g) \mapsto g\). We define a group structure on \( Y \) by

\[
(g,u) \cdot (h,v) = (gh,g \cdot v + u).
\]

Then one can check that this is in fact a crossed extension.

The idea is that extensions of \( G \) by \( A \) is the same as a crossed homomorphism \( G \to BA \), which is the same as a homomorphism \( I \to BA \).

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Today we will take a look at the particular case of group cohomology in which our group $G$ of interest is the Galois group of a field extension. The hope is that this gives us some information about the field extensions in question.

**Definition 37.** Let $L$ be a field, let $G$ be a group acting on $L$ (on the left). A $G-L$-vector space is an $L$-vector space with a $G$-module structure such that

$$g \cdot (\lambda x) = g \cdot (\lambda)g \cdot (x), \quad \forall g \in G, \lambda \in L, x \in V.$$

**Proposition 24.** Let $L$ be a Galois extension of $K$, $G = \text{Gal}(L/K)$. Then $H^1(G, L^*)$ is isomorphic to the set of 1-dimensional $G-L$-vector spaces modulo isomorphism, where $L^* = L \setminus \{0\}$ is the multiplicative group of $L$.

**Proof.** Note that this amounts to procuding an appropriate action on $L$ by $G$. Recall that $H^1(G, L^*) = \text{Der}(G, L^*)/\text{PDer}(G, L^*)$. So given a crossed homomorphism $\sigma$, we must produce a $G$-action on $L$.

TODO: missed what this action was.

**Theorem 19.** Suppose $K \subset L$ is a Galois Extension with Galois group $G$. Then there is an equivalence of categories

$$K\text{-Vect} \rightarrow G-L\text{-Vect}$$

sending

$$V \mapsto L \otimes_K V,$$

where the action is given by $g \cdot (a \otimes x) = (g \cdot a) \otimes x$.

We will come back to the proof.

**Corollary 9.**

$$H^1(G, L^*) = 0.$$

**Proof.** Suppose $L'$ is a 1-dimensional $G-L$-vector space, then $L' \cong L \otimes_K K$ since $K$ is the unique 1-dimensional $K$-vector space $L' \cong L$ as $G-L$-vector spaces.

**Definition 38.** A $K$-algebra $A$ is $L$-split if and only if

$$L \otimes_K A \cong L^n$$

as an $L$-algebra.

As an example, let $K \subset L$ be a Galois extension of the form $L = K[x]/f(x)$. Then

$$L \otimes_K L = L[x]/f(x) = L[x]/\prod_i (x - \alpha_i) = \prod_i L[x]/(x - \alpha) = \prod_i L$$

where the correspondence sends $x \in L[x]/f(x)$ to $(\alpha_i)_i$.

**Remark 10.** The theorem above is the fundamental theorem of Galois Theory. The equivalence in the theorem preserves the tensor product operation since it is given by tensor products and tensor products are associative. But the tensor product is used to define algebras, so we also get an equivalence of categories on the level of algebras:

$$K\text{-algebras} \sim G-L\text{-algebras}.$$
Similarly, we get a correspondence

$$(L\text{-split } K\text{-algebras}) \rightarrow (L\text{-split } G\text{-L-algebras}) \rightarrow (\mathcal{G}^{op} \text{- sets})$$

where the second map above sends

$$A \rightarrow \text{Hom}_K(A, L).$$

Note then that

$$L \rightarrow \text{Hom}_K(L, L).$$

We now prove Theorem 19.

**Proof.** The inverse equivalence will be $W \rightarrow W^G$. We need to show that there are two isomorphisms

$$V \sim (L \otimes_K V)^G$$

and

$$L \otimes_K W^G \sim W.$$

To prove the first one, it is sufficient to show that

$$L \otimes V \rightarrow L \otimes (L \otimes V)^G$$

is an isomorphism, where all tensor products are over $K$ unless otherwise decorated. $\square$

<++>

**Lecture 30: ∞-Categories - 04/27/2020**

Infinity categories are a bit of a slippery thing. People have different models for what they are. There are methods for talking about them independently of the model, but things get very abstract very fast. Instead, we will follow Lurie and take an approach via quasicategories. Usually one describes a category by specifying objects and morphisms, then checking that composition works correctly. The problem comes when we start trying to build categories like the homotopy category of chain complexes in an abelian category. Here composition is not necessarily defined on the nose, but defined up to homotopy. A good example to keep in mind visually is the fundamental group of a space $X$:

$$
\pi_1(X) = [S^1 \rightarrow X]/\sim,
$$

where the quotient is by homotopy equivalence. The group operation is by concatenation. To prove that this is indeed a group one needs to scale the domain.

The prototype infinity category is the fundamental groupoid of $X$, denoted $\Pi(X)$. This is a category whose objects are the points of $X$ and the morphisms $\text{Hom}(x, y)$ for $x, y \in X$ are the paths from $x$ to $y$. The approach of quasicategories says the following: instead of remembering morphisms and composition laws, remember all commutative triangles. More explicitly, given a sequence of morphisms below

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{h} \\
C & & D
\end{array}
$$

where the second map above sends

$$A \rightarrow \text{Hom}_K(A, L).$$

Note then that

$$L \rightarrow \text{Hom}_K(L, L).$$

We now prove Theorem 19.

**Proof.** The inverse equivalence will be $W \rightarrow W^G$. We need to show that there are two isomorphisms

$$V \sim (L \otimes_K V)^G$$

and

$$L \otimes_K W^G \sim W.$$
Instead of saying there is a single \( f * g : A \to C \) making the diagram commute, we keep track of all possible arrows \( A \to C \) which do this. The way a quasicategory does this is by having, not only a set of objects \( \mathcal{C}_0 \) and a set of morphisms \( \mathcal{C}_1 \), but a set of commuting triangles \( \mathcal{C}_2 \), a set of commuting tetrahedra \( \mathcal{C}_3 \), and so on. If you've seen simplicial sets before, then you know where this is going.

**Definition 39.** \( \Delta \) will denote the category whose objects are finite ordered sets \( [n] = \{0, \ldots, n\} \) and morphisms non-decreasing maps \( [n] \to [m] \). A simplicial set \( \mathcal{C} \) is a functor \( \mathcal{C} : \Delta^{op} \to \text{Set} \).

It is useful to think of \( \mathcal{C} \) as a recipe for how to glue simplices together. In fact, this is made explicit by geometric realization which is a functor from the category of simplicial sets to topological spaces. Now it seems like we're getting what we want out of this definition, however, we are not quite done.

For each \( n \), let \( \Delta^n \) be the category depicted below:

\[
\begin{array}{ccc}
0 & \rightarrow & 1 & \rightarrow \cdots & \rightarrow & n-1 & \rightarrow & n
\end{array}
\]

If \( \mathcal{C} \) is a category, let \( \mathcal{C}_n = \text{Hom}(\Delta^n, \mathcal{C}) \). Note then that \( \mathcal{C}_0 \) is the set of objects and \( \mathcal{C}_1 \) the set of morphisms in \( \mathcal{C} \). Continuing on, we see that \( \mathcal{C}_2 \) is the set of pairs of composable arrows

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f} & X_1 & \xrightarrow{g} & X_2
\end{array}
\]

in \( \mathcal{C} \), and \( \mathcal{C}_3 \) is the set of triples of composable arrows

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f} & X_1 & \xrightarrow{g} & X_2 & \xrightarrow{h} & X_3
\end{array}
\]

**Remark 11.** Note that each \( \mathcal{C}_n \) is determined by its predecessors. As an example, we can think of \( \mathcal{C}_2 \) as a sort of fiber-product:

\[
\mathcal{C}_2 = \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1,
\]

where the fiber-product encodes the condition that the target of the first factor in \( \mathcal{C}_1 \) be equal to the source of the second factor \( \mathcal{C}_1 \).

The key here is to notice that the diagrams above can be filled in for any category. Looking at the \( f - g \) diagram above, we can 'fill in the triangle' by adding \( g \circ f : X_0 \to X_2 \). Similarly, in the \( f - g - h \) diagram above, we can fill in more:

\[
\begin{align*}
g \circ f : X_0 & \to X_2, \\
h \circ g : X_2 & \to X_3, \\
h \circ (g \circ f) : X_0 & \to X_3
\end{align*}
\]

Adding these in amounts to 'filling in the tetrahedron'. This filling condition is precisely what our definition is missing.

**Definition 40.** The category \( \Delta^n \) represents the simplicial set \( \text{Hom}_\Delta(\underline{\_}, [n]) \). We call this the standard \( n \)-simplex. The \( i \)-th horn of \( \Delta^n \), denoted \( \Lambda^n_i \) is obtained by omitting the index \( i \) in the definition of \( \Delta^n \).

**Definition 41.** A simplicial set \( \mathcal{C} \) is called a quasicategory (or \( \infty \)-category) if

\[
\text{Hom}(\Delta^n, \mathcal{C}) \twoheadrightarrow \text{Hom}(\Lambda^n_i, \mathcal{C})
\]

is surjective for all \( n, 0 < i < n \).
Remark 12. Here we view $\Delta^n$ both as the category defined above, and the functor that it represents, namely $\text{Hom}_\Delta(_, [n])$.

We note that a slight variant of the definition of a quasicategory above is often of interest. If we additionally require the same condition to hold for $i = 0, n$, the resulting notion is called a Kan complex ($\infty$-groupoid).

Example 19. Infinity category of spaces:

$$\mathcal{C}_0 = \text{Ob}(\text{Top})$$

$$\mathcal{C}_1 = \{X_0 \overset{f_0}{\longrightarrow} X_1\}$$

$$\mathcal{C}_2 = \left\{ \begin{array}{c} X_1 \\ \downarrow f \\ X_0 \quad \overset{g}{\underset{h}{\longrightarrow}} \quad X_2 \end{array} \right\} \text{commutes up to homotopy}$$

Lecture 31: $\infty$-Categories (continued) - 04/29/2020

Let’s start with a summary of what we learned last time.

Definition 42. An $\infty$-category is a simplicial set $\mathcal{C}$ such that

$$\text{Hom}(\Delta^n, \mathcal{C}) \twoheadrightarrow \text{Hom}(\Lambda^n_i, \mathcal{C})$$

is surjective for all inner horns $\Lambda^n_i \subset \Delta^n$.

We called this a quasicategory last time, following Quillen. This definition encodes a composition law that is well-defined only up to contractible ambiguity.

Definition 43. An $\infty$-category $\mathcal{C}$ is called stable if

1. $\mathcal{C}$ has a zero object
2. $\mathcal{C}$ has fibers and cofibers
3. a sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

is left exact if and only if it is right exact.

Note the similarities between this and the definition of an abelian category. Note that stability is a property of $\mathcal{C}$, not an additional structure. For a comparison, take triangulated categories. These come with an additional structure: distinguished triangles, the shift operator, etc. So if we can do the same things with $\infty$-categories that we can do with triangulated categories, then we have made progress because the former is more efficient. It remains to say what all the things in the definition above mean. In an $\infty$-category $\mathcal{C}$, $X \in \mathcal{C}_0$ is initial if $\text{Hom}_\mathcal{C}(X, Y)$ is contractible for all $Y$ and final if $\text{Hom}_\mathcal{C}(Y, X)$ is contractible for all $Y$. Note that here we replace the notion of a single point in ordinary categories with the notion of contractibility, being homotopy equivalent to a point. Note by 'contractible' we also mean nonempty. Again, a zero object is an object that is both final and initial.
**Definition 44.** Given $(X 	o Y) \in \mathcal{C}_1$, we define the fiber of this morphism by

$$\text{fiber}(X \to Y) = X \times_Y 0.$$ 

That is, it is the universal object

\[
\begin{array}{ccc}
\text{fiber}(X \to Y) & \longrightarrow & X \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Y
\end{array}
\]

In terms of $\infty$-categories, this means that it is a final object of the slice category $\mathcal{C}/I$, where $I$ is the diagram:

$$0 \to Y \leftarrow X.$$

Cofibers are defined dually. The last condition in the definition of an infinity category is equivalent to saying that the square

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Z
\end{array}
\]

is a cofiber square if and only if it is a fiber square. Fun fact: Lurie calls this square a triangle.

**Theorem 20.** Let $\mathcal{C}$ be an abelian category. Then the nerve of $\mathcal{C}$, $N(\mathcal{C})$ is a stable infinity category.
Solutions to Exercises

Solution 1. Exercise 1
Let $M$ and $N$ be $R$-modules. We claim that the standard construction of the direct sum as the set $M \oplus N = \{(m,n) | m \in M, n \in N\}$ with addition and $R$-action defined component-wise, is in fact the product and coproduct of $M$ and $N$. To show this, we first note that there are canonical projections $\pi_M : M \oplus N \to M$ and $\pi_N : M \oplus N \to N$ given by $\pi_M(m,n) = m$ and $\pi_N(m,n) = n$. Likewise, we have canonical inclusions $\iota_M : M \to M \oplus N$ given by $\iota_M(m) = (m,0)$ and $\iota_N : N \to M \oplus N$ given by $\iota_N(n) = (0,n)$. Let $Z$ be an arbitrary $R$-module with $R$-module homomorphisms $Z \xrightarrow{f} M$ and $Z \xrightarrow{g} N$. We can then a map $h : Z \to M \oplus N$ by $h(z) = (f(z),g(z))$ so that the following diagram commutes

$$
\begin{array}{ccc}
Z & \xrightarrow{h} & M \oplus N \\
\downarrow{g} & & \downarrow{\pi_N} \\
M & \xrightarrow{\pi_M} & N
\end{array}
$$

The map $h$ is evidently the unique such map, for if $h' : Z \to M \oplus N$ were a map such that $h'(z) = (m,n)$ where $f(z) \neq m$ or $g(z) \neq n$ then either $\pi_N \circ h \neq g$ or $\pi_M \circ h \neq f$. Thus $M \oplus N$ is the product in the category of $R$-modules.

Now suppose $Z$ is an $R$-module with maps $f : M \to Z$ and $g : N \to Z$. Then we can define a map $h : M \oplus N \to Z$ by $h(m,n) = f(m)+g(n)$. Now for any $m \in M$, $h \circ \iota_M(m) = h(m,0) = f(m) + g(0) = f(m)$ and for any $n \in N$, $h \circ \iota_N(n) = h(0,n) = f(0) + g(n) = g(n)$. So the corresponding diagram for the coproduct commutes. Again, the map $h$ is the unique such map with this property. To see this, suppose $h' : M \oplus N \to Z$ satisfies $h' \circ \iota_M(m) = f(m)$ and $h' \circ \iota_N(n) = g(n)$ for all $m \in M$ and $n \in N$. Then

$$
h'(m,n) = h'(m,0) + h'(0,n) = h' \circ \iota_M(m) + h' \circ \iota_N(n) = f(m) + g(n).
$$

Thus $M \oplus N$ is the coproduct in the category of $R$-modules.

That finite products and coproducts agree follows by a simple induction. This shows that $\textbf{R-mod}$ satisfies condition AB0.

For condition AB1, it is easy to see that the usual constructions $\ker(f : M \to N) = \{m \in M | f(m) = 0\}$ and $\coker(f : M \to N) = N/f(M)$ satisfy the required properties of the kernel and cokernel as defined in Lecture 1.

Similarly, one can see that $\text{im}(f) = \{n \in N | \exists m \in M \text{ s.t. } f(m) = n\}$ and $\text{coim}(f) = M/\ker(f)$, satisfy the required properties of image and coimage. That these coincide is a standard so-called ‘isomorphism theorem’.

Solution 2. Exercise 2
Assume conditions AB1 and AB2. Using universality of the coimage and the fact that the composition $\ker(f) \to X \xrightarrow{f} Y$ is 0 we obtain a unique morphism $h : \text{coim}(f) \to Y$ making the following diagram commute.

$$
\begin{array}{ccc}
\ker(f) & \xrightarrow{0} & X \\
\downarrow{\exists! h} & & \downarrow{f} \\
\text{coim}(f) & \xrightarrow{0} & \text{im}(f)
\end{array}
$$

$$
\begin{array}{ccc}
& & Y \\
\downarrow{0} & & \downarrow{0} \\
\text{coim}(f) & \xrightarrow{0} & \text{im}(f)
\end{array}
$$

Now we want to use the universal property of \( \text{im}(f) \), which amounts to showing that the composition \( \text{coim}(f) \xrightarrow{h} Y \rightarrow \text{coker}(f) \) is 0. We proceed in a manner precisely dual to the argument in Lecture 2. Let \( g : A \rightarrow B \) be any morphism and \( Z \) any object. Let \( \sigma : B \rightarrow \text{coker}(g) \) be the universal map. Then we have a map \( \text{Hom}(\text{coker}(g), Z) \rightarrow \text{Hom}(B, Z) \) given by \( (t : \text{coker}(g) \rightarrow Z) \mapsto t \circ \sigma \). Note that the image of this map consists of maps \( t' : B \rightarrow Z \) such that \( t' \circ g = 0 \). Then by the universal property of cokernels, to each such \( t' \), there is a unique \( t : \text{coker}(g) \rightarrow Z \) such that \( t' = t \circ \sigma \). This shows that the map on Hom-sets is in fact injective. In particular, if \( t \circ \sigma = 0 \) then \( t = 0 \). Viewing the coimage \( \text{coim}(f) \) above as a cokernel then since the composition \( X \rightarrow \text{coim}(f) \xrightarrow{h} Y \rightarrow \text{coker}(f) \) is zero, we must have that \( \text{coim}(f) \xrightarrow{h} Y \rightarrow \text{coker}(f) \) is zero, as desired. This gives us our map \( \text{coim}(f) \rightarrow \text{im}(f) \).

**Solution 3. Exercise 3**

First we prove that the composition of the maps we denote by matrices can be computed via matrix multiplication at least in the case that they are diagonals. Let \( A, B, C, \) for \( i = 1, 2 \) be objects in an abelian category \( \mathcal{A} \). Let \( f_i : A_i \rightarrow B_i \) and \( g_i : B_i \rightarrow C_i \) be morphisms. Now consider the following diagram.

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & B_1 \\
\downarrow & & \downarrow \quad (g_1)_{(0)} \\
A_1 \oplus A_2 & \xrightarrow{M_f} & B_1 \oplus B_2 \\
\uparrow & \uparrow \quad (f_2)_{(0_2)} & \uparrow \quad (g_2)_{(0_2)} \\
A_2 & \xrightarrow{f_2} & B_2 \\
& \uparrow \quad g_2 & \uparrow \quad g_2 \\
& C_2 \\
\end{array}
\]

Here

\[
M_f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}, \quad \text{and} \quad M_g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}.
\]

Now the outer maps \( g_i \circ f_i : A_i \rightarrow C_i \) induce a unique map

\[
\begin{pmatrix} g_1 \circ f_1 & 0 \\ 0 & g_2 \circ f_2 \end{pmatrix} : A_1 \oplus A_2 \rightarrow C_1 \oplus C_2,
\]

which makes the diagram commute. However, the map \( M_g \circ M_f \) is seen the make the diagram commute, so indeed

\[
\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} = \begin{pmatrix} g_1 \circ f_1 & 0 \\ 0 & g_2 \circ f_2 \end{pmatrix}.
\]

Now the proof of associativity follows from a diagram chase of the following diagram. Here we denote a two by two diagonal matrix with \( f \) and \( g \) along the diagonal, for example, by \( M_{f,g} \). One can see that the bottom-most and top-most paths \( X \rightarrow Y \) correspond to \( (f + g) + h \) and \( f + (g + h) \), respectively. The two quarter-circle shaped diagrams with blue arrows commute by what was just proven, and the complements of each of these within their trapezoidal-shaped diagrams commute by associativity of the direct sum. Thus, the entire diagram commutes and the result follows.
A similar argument as the one above shows that matrix composition works as it should for off-diagonal matrices as well, that is,

\[
\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 0 & f_1 \\ f_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & g_1 \circ f_1 \\ g_2 \circ f_2 & 0 \end{pmatrix}.
\]

Using this, one can see that the following digram commutes, which in turn implies that the addition of functions as defined is commutative.

It remains to show that the zero map \( X \xrightarrow{0} Y \) is the identity with respect to this operation. To see this, consider the following diagram.

That the two triangles on the sides commute follows simply from the definition of the diagonal maps \( \Delta_X^* \) and \( \Delta_Y \). That the center square commutes follows from computing

\[
p_1^Y \circ (i_1^Y \circ f \circ p_1^X) \circ i_1^X = (p_1^Y \circ i_1^Y) \circ f \circ (p_1^X \circ i_1^X) = id_Y \circ f \circ id_X = f.
\]
Thus, the top path $X \to Y$ is the same as the bottom path $X \to Y$, implying $f + 0 = f$. By commutativity, we also have $0 + f = f$ so the zero map is indeed the identity with respect to this operation on $\text{Hom}(X, Y)$.

Solution 4. Exercise 4

That $1 \implies 2$ follows simply from the fact that exactness, by definition, means $\ker(d_n) = \text{im}(d_{n+1})$ for all $n$, so that

$$H_n(C) = \ker(d_n)/\text{im}(d_{n+1}) = 0.$$ 

Similarly, by definition, for all $n$, we have

$$H_n(C) = \ker(d_n)/\text{im}(d_{n+1}).$$

So if $H_n(C) = 0$ for all $n$, then $C$ must be exact. Thus $2 \implies 1$. Evidently $H_n(0) = 0$ for all $n$, where the argument, 0, of $H_n$ is the zero complex. Thus, if $C$ is acyclic so that $H_n(C) = 0$ for all $n$, then the unique map $H_n(0) \to H_n(C)$ must be an isomorphism $0 \to 0$, so $2 \implies 3$. Similarly, if the unique induced map $H_n(0) \to H_n(C)$ is an isomorphism then since the zero object is unique up to isomorphism, we must have $H_n(C) = 0$ for all $n$. Thus $3 \implies 2$, and the proof is complete.

Solution 5. Exercise 5

Let $R[x_1, \ldots, x_k]$ denote the free $R$-module on the generators $\{x_1, \ldots, x_k\}$. Since $d_0 = 0$ we have $\ker(d_0) = C_0 = R[v_1, \ldots, v_V]$. Now for any edge $e$, let $s(e), t(e) \in [v_1, \ldots, v_V]$ denote the source and target of the edge. Then the differential acts on $e$ by $d(e) = t(e) - s(e)$. Thus, the image of $d$ is generated by the differences of adjacent vertices. Since $\Gamma$ is connected, for any two vertices $v_i$ and $v_j$, there is a path connecting them, that is, there exists a sequence of vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_l}\}$ such that for each $k = 1, \ldots, l - 1$, there exists a vertex $e_k$ with either $s(e_k) = v_{i_k}$, $t(e_k) = v_{i_{k+1}}$ or $s(e_k) = v_{i_{k+1}}$, $t(e_k) = v_{i_k}$. Let $e_k = +1$ in the former case and $e_k = -1$ in the latter. Then

$$d\left(\sum_{k=1}^{l-1} e_k e_k\right) = \sum_{k=1}^{l-1} e_k d(e_k) = (v_{i_2} - v_{i_1}) + (v_{i_3} - v_{i_2}) + \cdots + (v_{i_l} - v_{i_{l-1}}) = v_{i_l} - v_{i_1}.$$ 

It follows that $[v_i] = [v_j]$ in $H_0(C)$ for any $i, j = 1, \ldots, V$. Moreover, since $\text{im}(d)$ is generated by differences of edges, $[v_i] \neq 0$ in $H_0(C)$ for any $i$. Thus

$$H_0(C) = R[v_1, \ldots, v_V]/(v_i - v_j) = R[e_1],$$

as desired.

Now since $d_2 = 0, H_1(C) = \ker(d)$. Evidently, if $(e_1, \ldots, e_l)$ is a closed path (i.e. $s(e_1) = t(e_l)$), then

$$d\left(\sum_{k=1}^{l} e_k e_l\right) = 0,$$

where $e_l = \pm 1$ depending on the orientation of each edge and the orientation on the cycle. Furthermore, if $\{e_i\}_{i=1}^{l}$ is a collection of disjoint edges, then

$$d\left(\sum_{i=1}^{l} r_i e_i\right) \neq 0,$$

for any $r_i \in R$. Thus, every element of $\ker(d)$, hence $H_1(C)$, is a sum of closed paths. Now let $T \subset \{e_1, \ldots, e_E\}$ be a minimal spanning tree for $\Gamma$. We claim that $|T| = V - 1$. An elementary proof can
be given by induction as follows. Suppose \( V = 1 \), then \( \emptyset \) is a spanning tree so we are done. Suppose now that the result holds for a graph with \( V - 1 \) vertices. Then for any graph with \( V \) vertices, we may take a subgraph of \( V - 1 \) vertices, find a spanning tree for this graph, then adding a single edge connecting the isolated point gives the result. Now, to each of the \( E - (V - 1) \) edges not in the spanning tree \( T \) of \( \Gamma \), we assign a closed path as follows. Let \( e \in \{e_1, \ldots, e_E\} \setminus T \). Starting at \( s(e) \) we trace a path whose first edge is \( e \), followed by the unique path \((t_1(e), \ldots, t_l(e))\) along the spanning tree back to \( s(e) \).

We claim that the elements\[
\hat{e} := e + \sum_{k=1}^{l} \epsilon_k t_k(e),
\]
for each \( e \in \{e_1, \ldots, e_E\} \setminus T \), form a free basis for \( \ker(d) \). It is evident that these are linearly independent, so it remains to show that any closed path is a sum of such elements. For this, let \((e_{i_1}, \ldots, e_{i_j})\) be a closed path. Then for at least one \( k \), \( e_{i_k} \in \{e_1, \ldots, e_E\} \setminus T \). Let \( e'_{i_1}, \ldots, e'_{i_j} \) be the edges in the complement of the spanning tree. Then\[
\sum_{m=1}^{j} \epsilon_m e'_{i_m} = \sum_{m=1}^{l} \epsilon_m e_{i_m},
\]
so we are done.

\textbf{Solution 10. Exercise 10}

Let \( f : B \to C \) be a chain map. We may view this as a double complex, \( D \), by using the sign trick. This complex is displayed below, where the extensions upwards and downwards are by zero objects and maps.

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots \\
\cdots & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
\vdots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B_{p+1} & \longrightarrow & B_{p} & \longrightarrow & B_{p-1} & \longrightarrow & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
C_{p+1} & \longrightarrow & C_{p} & \longrightarrow & C_{p-1} & \longrightarrow & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

We can then obtain a map of double chain complexes \( \delta : D \to B[0][+1] \), where the notation \( B[0][+1] \) means we shift the \( q \)-degree of \( B \) by \(+1\) (here \( B \) is viewed as a double chain complex concentrated in degree \( q = 0 \)), by simply applying the identity in degree \( q = 1 \) and the zero map elsewhere. Similarly, we obtain a map \( \iota : C \to D \) given by the identity in degree \( q = 0 \) and zero maps elsewhere. The resulting sequence is evidently exact.

\textbf{Solution 11. Exercise 11}

We take the liberty to assume the preceding theorem in Weibel, which states that a short exact sequence of chain complexes induces a long exact sequence in homology. It follows that the short exact sequence\[
0 \to A \to B \to C \to 0
\]
induces a long exact sequence
\[ \cdots \rightarrow H_{n+1}(C) \xrightarrow{\partial} H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots \]

By Exercise 4 above, if two of the three chain complexes are exact, we will have that each \( H_n \) for every \( n \) will be zero for those two chain complexes. The exact sequence above then implies that the remaining terms must be zero.

**Solution 12. Exercise 12**

Suppose we are given the following commutative diagram in an abelian category such that the columns are exact.

\[
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & A' & f' & B' & g' & C' & \rightarrow & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & A & f & B & g & C & \rightarrow & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & A'' & f'' & B'' & g'' & C'' & \rightarrow & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Suppose the bottom two rows are exact. Then, by commutativity, \( g \circ f \circ \alpha' = \gamma' \circ g' \circ f' \). By exactness of the middle row, \( g \circ f = 0 \), implying \( 0 = \gamma' \circ g' \circ f' \). However, by exactness of the rightmost column, \( \gamma' \) is monic, and so this implies \( g' \circ f' = 0 \). It follows that the top row is also a chain complex and so, by Exercise 11 above, it must be exact. Similarly, suppose the top two rows are exact. Then, by commutativity, \( g'' \circ f'' \circ \alpha = \gamma \circ g \circ f \). By exactness of the middle row, \( g \circ f = 0 \), implying \( g'' \circ f'' \circ \alpha = 0 \). However, by exactness of the leftmost column, \( \alpha \) is an epi and so this implies \( g'' \circ f'' = 0 \). It follows that the bottom row is also a chain complex and so, by Exercise 11 above, it must be exact.

Now suppose the top and the bottom row are exact and that \( g \circ f = 0 \). Then each row is automatically a chain complex and so the result, again, follows from the preceding exercise.

**Solution 13. Exercise 13**

Let \( f : C \rightarrow D \) be a chain map. We then have the following two short exact sequences
\[ 0 \rightarrow \ker(f) \rightarrow C \rightarrow \text{coim}(f) \rightarrow 0 \]
\[ 0 \rightarrow \text{im}(f) \rightarrow D \rightarrow \text{coker}(f) \rightarrow 0. \]

We then have the following two long exact sequences
\[
\cdots \rightarrow H_n(\ker(f)) \rightarrow H_n(C) \rightarrow H_n(\text{coim}(f)) \rightarrow H_{n-1}(\ker(f)) \rightarrow \cdots \\
\cdots \rightarrow H_{n+1}(\text{coker}(f)) \rightarrow H_n(\text{im}(f)) \rightarrow H_n(D) \rightarrow H_n(\text{coker}(f)) \rightarrow \cdots.
\]

Assuming \( \ker(f) \) and \( \text{coker}(f) \) are acyclic, these sequences give us isomorphisms \( H_n(C) \xrightarrow{\cong} H_n(\text{coim}(f)) \), \( H_n(\text{im}(f)) \cong H_n(D) \). Then since we are in an abelian category we have a canonical isomorphism \( \text{coim}(f) \rightarrow \text{im}(f) \), the composition
\[ H_n(C) \xrightarrow{\cong} H_n(\text{coim}(f)) \xrightarrow{\cong} H_n(\text{coim}(f)) \xrightarrow{\cong} H_n(D). \]
is then an isomorphism. Further, by definition of this canonical isomorphism between coimages and images, the diagram below commutes.

\[
\begin{array}{ccc}
\ker(f) & \xrightarrow{f} & C \\
\uparrow & & \downarrow \\
\text{coim}(f) & \xrightarrow{\cong} & \text{im}(f)
\end{array}
\]

Therefore, this isomorphism is indeed the map induced by \( f \), and so \( f \) is a quasi-isomorphism.

The converse is not true in general. To see this, consider the following chain map as a counter example.

\[
\begin{array}{cccccc}
0 & \xrightarrow{0} & 0 & \xrightarrow{1} & Z & \xrightarrow{1} & Z & \xrightarrow{0} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{1} & Z & \xrightarrow{1} & 0 & \xrightarrow{0} & 0
\end{array}
\]

The kernel of this chain map is 
\( 0 \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \),
which is not acyclic. However, one can easily check that this is indeed a quasi-isomorphism.

**Solution 14. Exercise 14**

That this is a chain complex follows simply from the computation
\[
d(d(f))_m = d \circ d(f)_m + (-1)^{n-2}d(f)_{m-1} \circ d \\
= d \circ (d \circ f_m + (-1)^{n-1}f_{m-1} \circ d) + (-1)^{n-2}d \circ f_{m-1} + (-1)^{n-1}f_{m-2} \circ d \circ d \\
= (-1)^{n-1}d \circ f_{m-1} \circ d + (-1)^{n-2}d \circ f_{m-1} \circ d = 0.
\]

As mentioned in the notes, the tricky part is that we do not notationally distinguish between the \( d \) being defined on the complex \( \text{Hom}(A, B) \) and the \( d \) of the chain complexes \( A \) and \( B \) themselves.

Now to compute the kernel of \( d_0 \), compute
\[
d_0(f)_m = d \circ f_m - f_{m-1} \circ d : A_m \rightarrow B_{m-1},
\]
but since \( f \in \text{Hom}_0(A, B) \) and \( \text{Hom}_0(A, B) \) is exactly the set of chain maps \( A \rightarrow B \), we have that this must be zero. Thus, \( Z_0 = \text{Hom}_0(A, B) \).

**Solution 15. Exercise 15** First we wish to show that acyclic bounded below chain complexes of free \( R \)-modules are always split exact. Let \( C \) be such a complex. Since we are given that \( C \) is acyclic it suffices to show that it is split. Since it is bounded below, we may assume that the last nonzero term is \( C_0 \). Then the end of this sequence looks like
\[
\cdots \rightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \rightarrow 0.
\]

Since \( C \) is acyclic, this sequence is exact and so \( d_1 \) must be surjective. Then, since \( C_0 \) is free we can choose a generating set \( \{t_a\} \) and preimages \( \{\hat{t}_a\} \) under \( d_1 \) so that we get a map \( s_0 : C_0 \rightarrow C_1 \) by mapping \( t_a \rightarrow \hat{t}_a \) and extending linearly. Evidently, \( d_1 \circ s_0 = 1 \implies d_1 \circ s_0 \circ d_1 = d_1 \) so we have defined the splitting map at the first stage. To define \( s_1 \), note that we have an exact sequence
\[
0 \rightarrow \ker(d_1) \rightarrow C_1 \rightarrow C_0 \rightarrow 0,
\]

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which splits since all modules in sight are free. Thus \( C_1 \cong \ker(d_1) \oplus C_0 \). Now since \( C \) is acyclic, we have \( \ker(d_1) \cong \text{im}(d_2) \). Furthermore, since this is a direct summand of a projective module we have the following diagram which shows the existence of a morphism \( \text{im}(d_2) \xrightarrow{s'_1} C_2 \).

\[
\begin{array}{ccc}
C_2 & \xrightarrow{d_2} & \text{im}(d_2) \\
\downarrow & & \uparrow \exists s'_1 \\
\text{im}(d_2) & \xrightarrow{1} & 0
\end{array}
\]

We then have a map \( s_1 = (s'_1 \ 0) : \text{im}(d_2) \oplus C_0 = C_1 \rightarrow C_2 \). That this is indeed a splitting map follows from the commutativity of the triangle above, that is,

\[
\hat{d}_2 \circ s'_1 = 1 \implies \hat{d}_2 \circ s'_1 \circ d_2 = 1.
\]

Continuing in this way, at the \( n \)-th step we have the exact sequence

\[
0 \rightarrow \ker(d_n) \rightarrow C_n \rightarrow \text{coker}(d_n) \rightarrow 0,
\]

which gives the splitting \( C_n = \ker(d_n) \oplus \text{im}(d_n) \cong \text{im}(d_{n+1}) \oplus \text{im}(d_n) \). From here we can proceed as above to obtain a splitting map \( s_n : C_n \rightarrow C_{n+1} \).

Now let \( C \) is an acyclic chain complex of finitely generated abelian groups, not necessarily bounded below. In this case, we still have the following exact sequence

\[
0 \rightarrow \ker(d_n) \rightarrow C_n \rightarrow \text{coker}(d_n) \rightarrow 0.
\]

Now since \( \mathbb{Z} \) is a PID, and \( \text{im}(d_n) \) is a submodule of the free \( \mathbb{Z} \)-module \( C_{n-1} \), we have that \( \text{im}(d_n) \) itself is free. So we get that the sequence above splits and the process is the same as above to obtain splitting maps.

**Solution 16. Exercise 16**

Let \( C \) be a split complex. Consider the associated short exact sequence at the \( n \)-th stage

\[
0 \rightarrow \ker(d_n) \rightarrow C_n \rightarrow \text{coker}(d_n) \rightarrow 0.
\]

The splitting condition \( d = dsd \) implies that this sequence is split so that

\[
C_n = \ker(d_n) \oplus \text{coker}(d_n) = \mathbb{Z}_n \oplus B'_{n},
\]

as desired. Similarly, since \( d^2 = 0 \), we have that

\[
0 \rightarrow \text{im}(d_{n+1}) \xrightarrow{d} \ker(d_n) \rightarrow \text{coker}(d) \rightarrow 0
\]

is an exact sequence. The splitting condition \( d = dsd \) again gives us that

\[
\ker(d_n) = \mathbb{Z}_n = \text{im}(d_{n+1}) \oplus \text{coker}(d) = B_n \oplus H'_n,
\]

as desired. Conversely, suppose we have such splittings. Then \( C_n = B_n \oplus H'_n \oplus B'_n \). However, \( B'_n \cong B_{n-1} \). Then \( C_{n+1} = B_{n+1} \oplus H'_{n+1} \oplus B_n \), so that we get a map

\[
s_n = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} : C_n = B_n \oplus H'_n \oplus B'_n \rightarrow C_{n+1} = B_{n+1} \oplus H'_{n+1} \oplus B_n.
\]
In terms of this decomposition, we also have
\[
d_n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]
so that \(d_{n+1} = d_{n+1}s_n d_{n+1}\), implying \(s_n\) is indeed a splitting map.

**Solution 17.** Exercise 17

Let \(C\) be a split exact chain complex. Then, by the previous exercise, we have that \(C_n = B_n \oplus B_{n-1}\). The splitting map is then given by
\[
s_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : C_n \to C_{n+1}
\]
and the differential by
\[
d_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : C_n \to C_{n-1}.
\]

Then
\[
d_{n+1} \circ s_n + s_{n-1} \circ d_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{C_n}.
\]
So the identity map is nullhomotopic.

**Solution 18.** Exercise 18

Let \(C\) be a split complex and \(H\) the complex consisting of \(H_n(C)\) at the \(n\)-th stage with zero maps as differentials. Now, by Exercise 1.4.2, since \(C\) is split, we have a decomposition \(C_n = Z_n \oplus B'_n\) where \(Z_n = \ker(d_n)\) and \(B'_n = \coker(d_n)\) for all \(n\). Furthermore, we have \(Z_n = B_n \oplus H_n\). Thus, we have the maps
\[
f_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : C_n = B_n \oplus H_n \oplus B'_n \to H_n
\]
and
\[
g_n = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : H_n \to C_n.
\]

Evidently \(f_n \circ g_n = 1 : H_n \to C_n\), but
\[
g_n \circ f_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

**TODO: complete solution.**

**Solution 19.** Exercise 19

First we show that chain homotopy equivalence defines an equivalence relation on \(\text{Hom}(C,D)\), the set of chain maps. Taking \(s_n = 0\) for all \(n\) we see that \(f - f = 0 = d \circ 0 + 0 \circ d\). So this relation is reflexive. Suppose, \(f - g = ds + sd\). Then \(g - f = d(-s) + (-s)d\), so this relation is symmetric. Suppose, additionally, that \(g - h = ds' + s'd\), then
\[
f - h = (f - g) + (g - h) = d(s + s') + (s + s')d.
\]
This shows that the relation is transitive so that this is indeed an equivalence relation. Let \(\text{Hom}_{K}(C,D)\) denote the set of equivalence classes. This inherits an abelian group structure from the abelian
is a chain map. Then start by defining its homotopy-inverse, \( \beta \) isomorphism. We will see in this exercise that this is in fact a chain homotopy equivalence. Let us

\[
\begin{align*}
\text{Exercise 21} & \quad \text{Solution 21.}
\end{align*}
\]

Thus, \( vf u - vgu = v(f - g)u = v(ds + sd)u = vdsu + vsd u = d(usu) + (usu)d. \]

So the maps imply chain homotopy equivalence. It follows that we can compose equivalence classes of chain maps and so there is a category \( K \) whose objects are chain complexes and morphisms are chain-homotopy equivalence classes of chain maps.

Now suppose \( f, g : C \to D \) for \( i = 1, 2 \) are chain maps. Let \( f - g = ds + sd \). Then

\[
\begin{align*}
(f_1 + f_2) - (g_1 + g_2) &= (f_1 - g_1) + (f_2 - g_2) = ds + s^1 d + ds^2 + s^2 d = d(s^1 + s^2) + (s^1 + s^2) d.
\end{align*}
\]

Conversely, suppose \( f, g : C \to D \). Then \( f - g = ds + sd \) is a chain map. Consider the mapping cylinder \( \text{cyl}(f) \) of \( f \). Let \( \alpha : C \to \text{cyl}(f) \) be defined in each degree by \( \alpha_n(c) = (0, 0, c) \). It was shown in Lemma 1.5.6 in Weibel that \( \alpha \) is a quasi-isomorphism. We will see in this exercise that this is in fact a chain homotopy equivalence. Let us start by defining its homotopy-inverse, \( \beta : \text{cyl}(f) \to C \) which is given in degree \( n \) by

\[
\beta_n(b, b', c) = f(b) + c.
\]

Solution 20. Exercise 20

Suppose first that \( f, g : C \to D \) are chain homotopic. Then we have maps \( s : C_n \to D_{n+1} \) such that

\[
f - g = ds + sd.
\]

Now consider the map

\[
\begin{align*}
\begin{pmatrix} f & s & g \end{pmatrix} & : C_n \oplus C_{n-1} \oplus C_n \to D_n.
\end{align*}
\]

That this is a chain map follows from the following computation

\[
\begin{align*}
\begin{pmatrix} f & s & g \end{pmatrix} \begin{pmatrix} d_C & 1_C & 0 \\
0 & -d_C & 0 \\
0 & -1_C & d_C \end{pmatrix} = (f \circ d_C, f - s \circ d_C - g \circ d_C) = (d_D \circ f, d_D \circ s, d_C \circ g) = d_D(f \circ s \circ g).
\end{align*}
\]

Conversely, suppose

\[
\begin{align*}
\begin{pmatrix} f & s & g \end{pmatrix} & : C_n \oplus C_{n-1} \oplus C_n \to D_n
\end{align*}
\]

is a chain map. Then

\[
\begin{align*}
\begin{pmatrix} f & s & g \end{pmatrix} \begin{pmatrix} d_C & 1_C & 0 \\
0 & -d_C & 0 \\
0 & -1_C & d_C \end{pmatrix} = d_D(f \circ s \circ g),
\end{align*}
\]

implying

\[
f - s \circ d_C - g = d_D \circ s \implies f - g = d_D \circ s + s \circ d_C.
\]

So the maps \( (0s0) : C_{n-1} \to D_n \) provide a chain homotopy between \( f \) and \( g \).

Solution 21. Exercise 21

Let \( f : B \to C \) be a chain map. Consider the mapping cylinder \( \text{cyl}(f) \) of \( f \). Let \( \alpha : C \to \text{cyl}(f) \) be defined in each degree by \( \alpha_n(c) = (0, 0, c) \). It was shown in Lemma 1.5.6 in Weibel that \( \alpha \) is a quasi-isomorphism. We will see in this exercise that this is in fact a chain homotopy equivalence. Let us start by defining its homotopy-inverse, \( \beta : \text{cyl}(f) \to C \) which is given in degree \( n \) by

\[
\beta_n(b, b', c) = f(b) + c.
\]
Now consider the map equivalence. Note that $h$:

$$h \circ \alpha(c) = h(0,0,c) = f(0) + c = c,$$

so $\beta \circ \alpha = 1_C$. Now

$$\alpha \circ \beta(b,b',c) = \alpha(f(b),c) = (0,0,f(b) + c),$$

so we don’t get an exact inverse in this direction. Consider the morphisms $s_n : \text{cyl}(f)_n \rightarrow \text{cyl}(f)_{n+1}$ given by $s(b,b',c) = (0,b,0)$, and compute

$$(ds + sd)(b,b',c) = d(0,b,0) + s(d(b) + b', -d(b'), d(c) - f(b')) = (b, -d(b), -f(b)) + (0,d(b) + b', 0) = (b, b', -f(b)) = (1 - \alpha \circ \beta)(b,b',c).$$

So $\alpha \circ \beta$ and $1 : \text{cyl}(f) \rightarrow \text{cyl}(f)$ are chain homotopic, implying $\alpha$ is a chain-homotopy equivalence.

**Solution 22.** Exercise 22

Let $f : B \rightarrow C$ be a chain map. Recall that the cone of $f$ is itself a chain complex with $\text{cone}(f)_n = B_{n-1} \oplus C_n$ and differential

$$d_f = \begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix}. $$

Now let $v : C \rightarrow \text{cone}(f)$ be the inclusion sending $c \mapsto (0,c)$. Then the cone of this inclusion is itself a chain complex with

$$\text{cone}(v)_n = C_{n-1} \oplus \text{cone}(f)_n = C_{n-1} \oplus B_{n-1} \oplus C_n,$$

and differential given by

$$d = \begin{pmatrix} -d_C & 0 \\ v & d_f \end{pmatrix} = \begin{pmatrix} -d_C & 0 & 0 \\ 0 & -d_B & 0 \\ -1_C & -f & d_C \end{pmatrix}. $$

Our goal is to exhibit a chain homotopy equivalence between this chain complex and the shifted complex $B[-1]$. We define maps

$$h_n : \text{cone}(v)_n \rightarrow B[-1]_n, \quad \text{by} \quad (c,b,c') \mapsto (-1)^n b,$$

and

$$g_n : B[-1]_n \rightarrow \text{cone}(f)_n, \quad \text{by} \quad b \mapsto ((-1)^{n+1} f(b), (-1)^n b, 0).$$

Note that $h_n \circ g_n = 1_B$ so we are halfway to showing that these maps indeed form a chain homotopy equivalence. Note that

$$(1_{\text{cone}(v)} - g_n \circ h_n)(c,b,c') = (c,b,c') - ((-1)^{n+1} f(b), b, 0) = (c + f(b), 0, c').$$

Now consider the map

$$s_n : C_{n-1} \oplus B_{n-1} \oplus C_n \rightarrow C_n \oplus B_n \oplus C_{n+1}, \quad \text{given by} \quad (c,b,c') \mapsto (-c', 0, 0).$$
This may be written as a $3 \times 3$ matrix with the only nonzero entry being $-1C_n$ on the top right entry. Then

$$\begin{pmatrix} -d_C & 0 & 0 \\ 0 & -d_B & 0 \\ -1C & -f & d_C \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & -d_B & 0 \\ -1C & -f & d_C \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & d_C \\ 0 & 0 & 0 \\ 0 & 1C \end{pmatrix} + \begin{pmatrix} 1C & f & -d_C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1C & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1C \end{pmatrix}.$$

This implies

$$(ds + sd)(c, b, c') = (c + f(b), 0, c') = (1_{\text{cone}(f)} - g_n \circ h_n)(c, b, c'),$$

so $g_n \circ h_n$ is chain homotopy equivalent to the identity and completing the proof.

**Solution 23. Exercise 23**

We invoke Proposition 6 in these notes which states that an abelian group is projective if and only if it is the direct summand of a free abelian group. Then since free subgroups of free abelian groups are free, it follows that an abelian group is projective if and only if it is free.

For this next part we show the result holds in general for $R$-modules, where $R$ is a PID. First suppose $I$ is an injective $R$-module. For any nonzero $r \in R$, the multiplication by $r$ map $R \to R$ is injective since PID’s are integral domains, and hence have no zero divisors. Now fix $x \in I$ and let $f : R \to I$ be given by $f(s) = sx$. Then we have the following diagram where injectivity of $I$ gives us the dashed map.

$$\begin{array}{ccc}
0 & \longrightarrow & R & \overset{r}{\longrightarrow} & R \\
& & \downarrow{f} & & \circlearrowleft {g} \\
& & I & & \\
\end{array}$$

Commutativity of the diagram then implies that $x = f(1) = \hat{f} \circ r(1) = \hat{f}(r) = r\hat{f}(1)$. It follows any $x \in I$ is of the form $ry$ for some $r \in R$ and $y \in I$, so $I$ must be divisible.

Conversely, suppose $I$ is divisible. By Baer’s Criterion in Weibel page 39, it suffices to check that we can extend maps from ideals $J \subset R$. Assuming that $R$ is a PID, we have $J = (r)$ for some $r \in R$. If $r = 0$ then we can simply extend by the zero map and be done. Assume, therefore, that $r \neq 0$. Suppose that $f : (r) \to I$ is an $R$-module homomorphism. Since $I$ is divisible, we have that there exists a $y \in I$ such that $f(r) = ry$. We can then define $\hat{f} : R \to I$ by $\hat{f}(1) = y$. This map is an extension since $f(r) = \hat{f}(r) = r\hat{f}(1) = ry$. Thus, $I$ must be injective.

**Solution 24. Exercise 24**

Let

$$I(A) = \prod_{f \in \text{Hom}_Z(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}.$$ 

We want to show that the canonical evaluation map

$$e_A : A \to I(A), \quad \text{given by } a \mapsto (f(a))_{f \in \text{Hom}_Z(A, \mathbb{Q}/\mathbb{Z})}$$
is injective. To do this, it suffices to find, for any \(a \in A\), a \(\mathbb{Z}\)-module homomorphism \(\hat{f} : A \to \mathbb{Q}/\mathbb{Z}\) such that \(\hat{f}(a) \neq 0\). First suppose, \(a \cdot n = 0\) for some \(n \in \mathbb{Z}\). Then we have a map
\[
f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, \quad \text{given by } a \mapsto \frac{1}{n}.
\]
Then since \(\mathbb{Q}/\mathbb{Z}\) is injective, we have the existence of an extension \(\hat{f}\) as in the following diagram.

\[
\begin{array}{ccc}
0 & \longrightarrow & a\mathbb{Z} \\
& & \downarrow f \\
& & A \\
& & \downarrow \kappa \\
& & \mathbb{Q}/\mathbb{Z} \\
\end{array}
\]

Then \(\hat{f} : A \to \mathbb{Q}/\mathbb{Z}\) is a map such that \(\hat{f}(a) = f(a) = \frac{1}{n} \neq 0\). Now if \(a \cdot n \neq 0\) for all \(n \in \mathbb{Z}\) then we can simply take \(f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}\) to by the map sending \(a\) to any nonzero element of \(\mathbb{Q}/\mathbb{Z}\) and the result still holds.

**Solution 25. Exercise 25**

Suppose we have the exact sequence below with \(P\) projective
\[
0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0.
\]
Then, since the derived functors are \(\delta\)-functors, we obtain from this the long exact sequence below
\[
\cdots \rightarrow L_n F(P) \rightarrow L_n F(A) \rightarrow L_{n-1} F(M) \rightarrow L_{n-1} F(P) \rightarrow \cdots
\]
Since \(P\) is projective, we have that \(L_n F(P) = 0\) for all \(n > 0\). Thus, \(L_n F(A) \cong L_{n-1} F(M)\) for all \(n \geq 2\). For \(n = 1\), we have that
\[
L_1 F(A) \cong L_0 F(M) = F(M) - L_0 F(P) = F(P)
\]
is exact so that \(L_1 F(A) = \ker(F(M) - F(P))\). More generally, suppose we have the exact sequence below with \(P_i\) projective
\[
0 \rightarrow M_m \rightarrow P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} P_0 \rightarrow A \rightarrow 0.
\]
We can break up this into short exact sequences as displayed in the diagram below.

\[
\begin{array}{ccc}
0 & \longrightarrow & M_m \\
& & \downarrow \im(f_m) \\
0 & \longrightarrow & P_m \\
& & \downarrow \\
& & \im(f_{m-1}) \\
& & \downarrow \\
& & 0
\end{array}
\]

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The horizontal row is exact and hence gives rise to a long exact sequence

\[ \cdots \to L_n F(P_m) \to L_n F(\text{im}(f_m)) \to L_{n-1} F(M_m) \to L_{n-1} F(P_m) \to \cdots \]

Since \( P_m \) is projective, \( L_n F(P_m) = 0 \) for all \( n > 0 \), this implies \( L_n F(\text{im}(f_m)) \cong L_{n-1} F(M_m) \) for \( n \geq 2 \).

The vertical column is also exact and so we also have the following long exact sequence

\[ \cdots \to L_n F(P_{m-1}) \to L_n F(\text{im}(f_{m-1})) \to L_{n-1} F(\text{im}(f_m)) \to L_{n-1} F(P_{m-1}) \to \cdots \]

Again, since \( P_{m-1} \) is projective, \( L_n F(P_{m-1}) = 0 \) for \( n > 0 \) so that \( L_n F(\text{im}(f_{m-1})) \cong L_{n-1} F(\text{im}(f_m)) \).

Putting these two together gives us that \( L_n F(\text{im}(f_{m-1})) \cong L_{n-2} F(M_m) \). Continuing in this way, we obtain

\[ L_n F(\text{im}(f_0)) = L_n F(A) \cong L_{n-1-m} F(M_m). \]

We note that this equation only holds for \( n \geq 2 + m \). For if \( n < 2 + m \) then the first isomorphism that we need, namely \( L_n F(\text{im}(f_m)) \cong L_{n-1} F(M_m) \), no longer holds. At the tail end of the first short exact sequence we have

\[ 0 \to L_1 F(\text{im}(f_m)) \to F(M_m) \to F(P_m) \to F(\text{im}(f_m)), \]

implying that \( L_1 F(\text{im}(f_m)) = \ker(F(M_m) \to F(P_m)). \) However, the above implies that \( L_1 F(\text{im}(f_m)) \cong L_{m+1} F(\text{im}(f_m)) = L_{m+1} F(A). \)

**Solution 26. Exercise 26**

For this exercise, we will show that the following are equivalent:

1. \( B \) is an injective \( R \)-module.
2. \( \text{Hom}_R(\_ , B) \) is an exact functor.
3. \( \text{Ext}^i(A, B) \) vanishes for all \( i \neq 0 \) and all \( A \).
4. \( \text{Ext}^1(A, B) \) vanishes for all \( A \).

\((1 \implies 2)\)

Suppose

\[ 0 \to X \to Y \to Z \to 0 \]

is an exact sequence of \( R \)-modules. We know that \( \text{Hom}_R(\_ , B) \) is left-exact, and so it suffices to check that the map

\[ \text{Hom}_R(Y, B) \to \text{Hom}_R(X, B) \]

induced by precomposition with \( X \to Y \) is surjective. This follows directly from \( B \) being injective as can be seen in the following diagram

\[
\begin{array}{c}
B \\
\uparrow \kappa \\
0 \longrightarrow X \longrightarrow Y
\end{array}
\]

It is evident that the reverse steps give the reverse implication. Explicitly, if \( \text{Hom}_R(\_ , B) \) is an exact functor then the induced map \( \text{Hom}_R(Y, B) \to \text{Hom}_R(X, B) \) is surjective, implying that for any solid-arrow diagram as above, the dashed map exists. so we also have that \((2 \implies 1)\).

\((1 \implies 3)\)

Assuming \( B \) is injective, the following is an injective resolution of \( B \)

\[ B \to 0 \to 0 \to \cdots \]
We denote this chain complex by $\hat{B}$. Now for any $R$-module $A$,

$$\text{Ext}^i_R(A,B) = R^i \text{Hom}_R(A,\_)(B) = H^i(\text{Hom}_R(A,\hat{B})).$$

The chain complex $\text{Hom}_R(A,\hat{B})$ is zero above and below degree zero and so its cohomology $H^i(\text{Hom}_R(A,\hat{B}))$ must vanish for all $i \neq 0$.

(3 $\implies$ 4)
This is a direct implication.

(4 $\implies$ 1)
Let $0 \to A' \to A$ be an exact sequence of $R$-modules. Then we have the short exact sequence

$$0 \to A' \to A \to A/A' \to 0$$

Since $\text{Ext}_R$ is a derived functor, it is a $\delta$-functor, and so there is an associated long exact sequence shown below.

$$\text{Ext}^0_R(A/A',B) \to \text{Ext}^0_R(A,B) \to \text{Ext}^0_R(A',B) \to \text{Ext}^1_R(A/A',B) \to \cdots$$

By assumption, $\text{Ext}^1_R(A/A',B) = 0$ so that $\text{Ext}^0_R(A,B) \to \text{Ext}^0_R(A',B)$ is surjective. However, since $\text{Ext}^0_R = \text{Hom}_R$, we have that for any solid arrow diagram below, there exists a dashed map. This completes the proof.

Note that here we have used the fact that $\text{Ext}^i_R(A,B)$ is also a $\delta$-functor on $A$.

**Solution 27. Exercise 27**

For this exercise, we will show that the following are equivalent:

1. $A$ is a projective $R$-module.
2. $\text{Hom}_R(A,\_)$ is an exact functor.
3. $\text{Ext}^i(A,B)$ vanishes for all $i \neq 0$ and all $B$.
4. $\text{Ext}^1(A,B)$ vanishes for all $B$.

(1 $\implies$ 2)
Assume $A$ is projective. Suppose

$$0 \to X \to Y \to Z \to 0$$

is an exact sequence of $R$-modules. We know that $\text{Hom}_R(A,\_)$ is left-exact, so it suffices to show that $\text{Hom}_R(A,Y) \to \text{Hom}_R(A,Z)$ is surjective. That is, given $f: A \to Z$, we want to exhibit a map $\tilde{f}: A \to Y$ such that the composition $A \xrightarrow{\tilde{f}} Y \to Z$ is $f$. It is evident that this is exactly the condition that $A$ be projective as one can see in the diagram below.

$$\begin{array}{c}
A \\
\downarrow f \\
Y \xrightarrow{\exists \tilde{f}} Z \xrightarrow{f} 0
\end{array}$$
As before, the logic walked in reverse shows that the implication (2 $\implies$ 1) also holds. Explicitly, if for any given solid-arrow diagram as above there exists the dashed arrow, then the map $\text{Hom}_R(A,Y) \rightarrow \text{Hom}_R(A,Z)$ is surjective so $\text{Hom}_R(A,\_)$ is exact.

(2 $\implies$ 3)
That $\text{Hom}_R(A,\_)$ is an exact functor implies that for any $R$-module $B$ and injective resolution,

$$B \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots,$$

the chain complex

$$\text{Hom}_R(A,B) \rightarrow \text{Hom}_R(A,I_0) \rightarrow \text{Hom}_R(A,I_1) \rightarrow \cdots$$

is exact. This is because injective resolutions are themselves exact sequences and the functor $\text{Hom}_R(A,\_)$ preserves exactness. It follows that the cohomology of this chain complex is zero for any nonzero degree. Since $\text{Ext}_i^R(A,B)$ is the $i$-th cohomology of this complex, we have that $\text{Ext}_i^R(A,B) = 0$ for all $i \neq 0$.

(3 $\implies$ 4)
This is a direct implication.

(4 $\implies$ 1).
Suppose $B \rightarrow B' \rightarrow 0$ is an exact sequence of $R$-modules. Then we have the following short exact sequence, where $K = \ker(B \rightarrow B')$,

$$0 \rightarrow K \rightarrow B \rightarrow B' \rightarrow 0.$$

Now according to Example 2.5.3 in Weibel, $\text{Ext}_i^R(A,B)$ can also be viewed as a right-derived contravariant functor in $B$. Thus, we have the following long exact sequence

$$\text{Ext}_R^1(A,K) \rightarrow \text{Ext}_R^0(A,B) \rightarrow \text{Ext}_R^0(A,B') \rightarrow \text{Ext}_R^1(A,K) \rightarrow \cdots$$

By assumption, $\text{Ext}_R^1(A,K) = 0$ so the map $\text{Ext}_R^0(A,B) \rightarrow \text{Ext}_R^0(A,B')$ is surjective. Since $\text{Ext}_R^0 = \text{Hom}_R$, we have that given a solid arrow diagram below, the dashed map exists. This completes the proof.

$$\begin{array}{ccc}
B & \longrightarrow & B' \\
\kappa & \uparrow & \\
A & \exists &
\end{array}$$

**Solution 28. Exercise 28**

Let $\_R = \{ r \in \mathbb{R} | rs = 0 \}$. Suppose $\_R \neq 0$ and consider the following exact sequence

$$0 \rightarrow \_R \rightarrow R \underset{r}{\rightarrow} R \rightarrow R/rR \rightarrow 0.$$ 

Note that since $R$ is a projective $R$-module, this is dimension shift by 1. Then by a direct application of Problem 25 above we have that

$$\text{Tor}^R_n(R/rR,B) \cong \text{Tor}^R_{n-2}(\_R,B)$$

for all $n \geq 3$. Now consider breaking up this sequence into two short exact sequences

$$0 \rightarrow \_R \rightarrow R \rightarrow \_R \rightarrow 0$$

and

$$0 \rightarrow \_R \rightarrow R \rightarrow R/rR \rightarrow 0.$$
We can now look at the Tor sequence of these sequences. Keeping in mind that \( R \) is projective, we have
\[
\cdots \to \text{Tor}_2^R(R, B) = 0 \to \text{Tor}_2^R(rR, B) \to \text{Tor}_1^R(rR, B) = 0 \to \text{Tor}_1^R(rR, B) \to B \to B \to rB
\]
and
\[
\cdots \to \text{Tor}_2^R(R, B) = 0 \to \text{Tor}_2^R(rR, B) \to \text{Tor}_1^R(rR, B) = 0 \to \text{Tor}_1^R(rR, B) \to rB \to B \to rB.
\]

**Solution 29. Exercise 29**

Let \( R \) be a commutative domain with field of fractions \( F \). Note that the quotient module \( F/R \) is generated by the elements \( \frac{1}{s} + R \in F/R \) for \( s \in R \). To see this, simply note that for \( \frac{\ell}{s} + R \in F/R \), we have \( r\left(\frac{1}{s} + R\right) = r\frac{1}{s} + R = \frac{\ell}{s} + R \). Thus,
\[
F/R = \lim_{s \in R} \left(\frac{1}{s} + R\right).
\]

Since Tor commutes with colimits, we have that
\[
\text{Tor}_1^R(F/R, B) = \text{Tor}_1^R(\lim_{s \in R} (\frac{1}{s} + R), B) = \lim_{s \in R} \text{Tor}_1^R((\frac{1}{s} + R), B).
\]

Now note that \( \frac{1}{s} + R \cong R/sR \) under the isomorphism \( \frac{1}{s} + R \to 1 \). Thus, by example 3.1.7 in Weibel which states that if \( r \) is not a zero divisor (a condition which is satisfied automatically since \( R \) is a domain) then \( \text{Tor}_1^R(R/rR, B) \cong R = \{ b \in B \mid rb = 0 \} \), we have that
\[
\text{Tor}_1^R(F/R, B) = \lim_{s \in R} B = \{ b \in B \mid 3s \in R, \text{ such that } sb = 0 \}.
\]

**Solution 30. Exercise 30**

**Solution 31. Exercise 31**

Suppose the following is an exact sequence of \( R \)-modules where both \( B \) and \( C \) are flat
\[
0 \to A \to B \to C \to 0.
\]

Then for every left \( R \)-module \( Z \), we have a long exact sequence
\[
\cdots \to \text{Tor}_2^R(Z, B) = 0 \to \text{Tor}_2^R(Z, C) = 0 \to \text{Tor}_1^R(Z, A) \to \text{Tor}_1^R(Z, B) = 0 \to \cdots.
\]

This implies \( \text{Tor}_1^R(Z, A) = 0 \) for all left \( R \)-modules \( Z \), implying \( A \) must also be flat.

**Solution 32. Exercise 32**

Let \( k \) be a field, \( R = k[x, y] \), and \( I = (x, y)R \). The point of this exercise is to show that \( I \) is a torsion-free ideal, but is not flat. Note that \( k = R/I \) so we have the following short exact sequence
\[
0 \to I \to R \to k \to 0.
\]

Applying Tor to this sequence gives us the following
\[
\cdots \to \text{Tor}_2^R(R, k) = 0 \to \text{Tor}_2^R(k, k) \to \text{Tor}_1^R(I, k) \to \text{Tor}_1^R(R, k) = 0 \to \cdots.
\]
This implies $\text{Tor}_2^R(k,k) \cong \text{Tor}_1^R(I,k)$. To show that $I$ is not flat, it thus suffices to show that $\text{Tor}_2^R(k,k)$ is nonzero. To this end, we take Weibel’s hint and consider the projective resolution of $k$ below

$$
0 \rightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \rightarrow k \rightarrow 0.
$$

Tensoring with $k$ and zooming in on degree 2, we find that $\text{Tor}_2^R(k,k)$ is the kernel of the map

$$
k \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} k^2.
$$

However since the action of $R$ on $k$ is given by multiplication by the constant term, both $y$ and $x$ act on $k$ by the zero map. Thus, the map above is in fact zero implying

$$
\text{Tor}_1^R(I,k) \cong \text{Tor}_2^R(k,k) = k.
$$

**Solution 33. Exercise 33**

Let $B^* = \text{Hom}_Z(B, \mathbb{Q}/\mathbb{Z})$ denote the Pontryagin dual of an abelian group $B$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, $\text{Hom}_Z(_, \mathbb{Q}/\mathbb{Z})$ is exact and so if

$$
A \rightarrow B \rightarrow C
$$

is an exact sequence of abelian groups then so is

$$
C^* \rightarrow B^* \rightarrow A^*.
$$

For the converse, we first note that by a direct consequence of Exercise 24 above we have that $B = 0$ if and only if $B^* = 0$. Now suppose $f : A \rightarrow B$ is an abelian group homomorphism such that $f^* : B^* \rightarrow A^*$ is the zero map. Consider the following commutative diagram with exact row and column.

$$
\begin{array}{ccc}
0 & \rightarrow & A^* \\
& & \downarrow f^*
\end{array}
\quad
\begin{array}{ccc}
im(f)^* & \rightarrow & 0 \\
& & \downarrow i^*
\end{array}
\quad
\begin{array}{ccc}
i^* & \rightarrow & B^*
\end{array}
$$

Then $\hat{f}^* \circ i^* = f^* = 0$, but $i^*$ is surjective, implying $\hat{f}^* = 0$. However, $\hat{f}^*$ is injective so we must have $\text{im}(f)^* = 0$ implying $\text{im}(f) = 0$ so that $f = 0$. Now suppose

$$
G^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*
$$

is exact. Since $(g \circ f)^* = f^* \circ g^* = 0$, we have that $g \circ f = 0$ so that $\text{im}(f) \subset \text{ker}(g)$. We now show that the inclusion $i : \text{im}(f) \rightarrow \ker(g)$ is surjective. Let $j : \ker(g) \hookrightarrow B$ be the canonical inclusion, $p : \ker(g) \rightarrow \ker(g)/\text{im}(f)$ the projection, and $\hat{f} : A \rightarrow \text{im}(f)$ the canonical map. Consider the following commutative diagram with exact rows and columns.
We will show that $i^*$ is injective. To this end, suppose $i^*(k) = 0$. Since $j^*$ is surjective, we have $b \in B^*$ such that $k = j^*(b)$. Since the diagram commutes, $(f')^* \circ i^* \circ j^*(b) = f^*(b) = 0$. By exactness of the dual sequence, there is $c \in C$ such that $g^*(c) = b$. Then $k = j^*(g^*(c)) = 0$ by exactness of the bottom row above. Thus, $i^*$ is injective, implying $(\ker(g)/\im(f))^* = 0$, which in turn implies $\ker(g)/\im(f) = 0$.

**Solution 34. Exercise 34**

Since $Z_{p^\infty} = Z[\frac{1}{p}] / Z$, there is an exact sequence

$$0 \to Z \to Z[\frac{1}{p}] \to Z_{p^\infty} \to 0.$$ 

This gives rise to a long exact sequence in Ext

$$0 \to \Hom_Z(Z_{p^\infty}, Z) \to \Hom_Z(Z[\frac{1}{p}], Z) \to \Ext^1_Z(Z, Z) \to \Ext^1_Z(Z_{p^\infty}, Z) \to \Ext^2_Z(Z[\frac{1}{p}], Z) \to \Ext^2_Z(Z, Z) = 0.$$ 

We claim that $\Hom_Z(Z[\frac{1}{p}], Z) = 0$. To see this suppose $f : Z[\frac{1}{p}] \to Z$ is an abelian group homomorphism and $f(1) = z$. Then for any $n \in \mathbb{N}$,

$$p^n f\left(\frac{1}{p^n}\right) = f\left(\frac{b^n}{p^n}\right) = f(1) = z.$$ 

This is only possible if $z = 0$, proving the claim. Thus,

$$\Ext^1_Z(Z[\frac{1}{p}], Z) \cong \Ext^1_Z(Z_{p^\infty}, Z) / \Hom_Z(Z, Z) \cong \hat{Z}_p / Z,$$

where the last isomorphism follows from Example 3.3.3 in Weibel and the fact that $\Hom_Z(Z, Z) \cong Z$.

**Solution 35. Exercise 35**

Let $p$ and $m$ be integers such that $p|m$. We first note that the following sequence is an injective resolution for $Z/p$ in the category of $Z/m$ modules.

$$0 \to Z/p \to Z/m \overset{p}{\to} Z/m \overset{m}{\to} Z/m \overset{p}{\to} Z/m \overset{m}{\to} \cdots.$$ 

Now let $A$ be an arbitrary $Z/p$-module. We are interested in computing $\Ext^n_{Z/m}(A, Z/m)$ for all $n$ in terms of $A^* = \Hom(A, Z/m)$. Applying the Hom-functor to this resolution gives us

$$0 \to \Hom_{Z/m}(A, Z/m) \overset{p}{\to} \Hom_{Z/m}(A, Z/m) \overset{m}{\to} \Hom_{Z/m}(A, Z/m) \overset{p}{\to} \Hom_{Z/m}(A, Z/m) \cdots.$$ 

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The cohomology is then computed to be

\[
\text{Ext}^n_{\mathbb{Z}/m}(A, \mathbb{Z}/p) = \begin{cases} 
p A^* & n = 0 \\
p \mathbb{A}^*/(p^*A^*) & n \text{ odd} \\
p \mathbb{A}^*/((m)^*A^*) & n \text{ even} \end{cases}
\]

Now let \( A = \mathbb{Z}/p \) and assume \( p^2 \mid m \). Then \( p^*A^* = 0 \), and \( \frac{m}{p} = kp \) for some \( k \in \mathbb{Z} \), implying \( (\frac{m}{p})^*A^* = 0 \) as well. Furthermore, since the zeroth derived functor returns the original functor, we have \( \text{Ext}^0_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/p) = \ker(p^*) = \text{Hom}_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/m) = \text{Hom}_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p \). Since each map in the complex above is zero in this case, we have that \( \text{Ext}^n_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p, \quad \forall n \).

**Solution 36. Exercise 36**

Note that \( \text{Ext}^1(M, N) \) is contravariant in \( M \) for fixed \( N \). Thus, to show that this is effaceable in both variables it suffices to show that \( \text{Ext}^1(M, N) = 0 \) if either \( N \) is injective or \( M \) is projective. Let \( X \) be an extension of \( M \) by \( N \). Then the diagram below shows that under either of these assumptions we get a splitting. Thus \( X \) is zero in \( \text{Ext}^1(M, N) \).

\[
\begin{array}{ccc}
0 & \rightarrow & N & \rightarrow & X & \rightarrow & M & \rightarrow & 0 \\
& & & \nearrow \quad \exists & & & \nearrow \quad \exists & \quad & \downarrow \\
& & N & \rightarrow & & & M & \rightarrow & 0
\end{array}
\]

While we're at it, let's go ahead and show that \( \text{Ext}^1(A, \_ \_ \_ \_) \) is in fact a \( \delta \)-functor in degrees \( \leq 1 \). Let \( 0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0 \)

be exact. Recall that in the notes, we defined the map \( \delta : \text{Hom}(A, B'') \rightarrow \text{Ext}^1(A, B') \) by pulling back sequences, namely, given \( \varphi : A \rightarrow B'' \) we pull back along \( B \rightarrow B'' \) to get an object \( X \) and a map \( X \rightarrow A \). Including \( B' \) into \( X \) gives us an extension

\[
0 \rightarrow B' \rightarrow X \rightarrow A \rightarrow 0.
\]

Suppose this sequence is zero, then we get a map \( A \rightarrow X \) giving us a map \( A \rightarrow B \) by composing with the map \( X \rightarrow B \) in the pullback square. This shows that \( \ker \delta \) is contained in the image of \( \text{Hom}(A, B) \rightarrow \text{Hom}(A, B'') \). A similar argument shows that the image of this map is also contained in \( \ker \delta \), implying that the long sequence we get in \( \text{Ext}^n \) for \( n \leq 1 \) is exact at that point.

**Solution 37. Exercise 37**

Let \( \{A_i\} \) be a tower of abelian groups in which all the maps \( A_{i+1} \rightarrow A_i \) are inclusions. We regard \( A = A_0 \) as a topological group where the open sets are sets of the form \( a + A_i \) for all \( a \in A \) and \( i \geq 0 \). We wish to show that \( \varprojlim A_i = \cap A_i = 0 \) if and only if \( A \) is Hausdorff under this topology. Let \( A \) be Hausdorff. Suppose, by way of contradiction, that \( \cap A_i \neq 0 \), and take \( a_0 \in \cap A_i \) to be nonzero. We view \( a_0 \) as an element of \( A \) under the given inclusions. Then any neighborhood of \( 0 \in A \) is of the form \( A_i \) for \( i \geq 0 \). But \( a_0 \in A_i \) for all \( i \geq 0 \) so \( a_0 \) and \( 0 \) cannot be separated by open sets in this topology contradicting our Hausdorff assumption. Conversely, suppose \( \cap A_i = 0 \). Then for any distinct \( a, b \in A \), there is an \( i \) such that \( a - b \notin A_i \). Now suppose there are \( a_i, a'_i \in A_i \) such that \( a + a_i = b + a'_i \), then \( a - b = a'_i - a_i \in A_i \), a contradiction. Thus, \( (a + A_i) \cap (b + A_i) = \emptyset \), implying \( A \) is Hausdorff.

Our next goal is to show that (if \( A \) is Hausdorff), then \( \varprojlim A = 0 \) if and only if \( A \) is complete in the sense that Cauchy sequences converge. A Cauchy sequence in a general topological abelian group \( A \)
is a sequence \( a_k \in A \) such that for all open neighborhoods \( U \) of 0, there exists \( N \) such that \( a_k - a_l \in U \) for all \( k, l > N \). In our case, since any open set around 0 is of the form \( A_i \) for some \( i \), a sequence \( a_k \) is Cauchy if for all \( i > 0 \) there is some \( N \) such that for all \( k, l > N, a_k - a_l \in A_i \). First, let us see what the assumption \( \lim^1 A_i = 0 \) tells us in the long exact sequence associated to

\[
0 \to \{A_i\} \to \{A\} \to \{A/A_i\} \to 0.
\]

From this, we obtain the following exact sequence.

\[
\lim A_i \longrightarrow A \overset{\varphi}{\longrightarrow} \lim A/A_i \overset{\delta}{\longrightarrow} \lim^1 A_i \longrightarrow 0
\]

Here we used the Mittag-Leffler condition and the fact that the maps \( A \to \) are surjective to obtain \( \lim^1 A = 0 \) on the right above. Assuming \( A \) is Hausdorff, \( \lim A_i = 0 \), as we have shown. In this case, it follows that \( \lim A/A_i \cong A \) if and only if \( \lim^1 A_i = 0 \). Thus, we are lured into showing that \( A \) is complete if and only if \( \lim A/A_i \cong A \). We turn to this now.

Suppose first that \( A \overset{\varphi}{\longrightarrow} \lim A/A_i \) is an isomorphism. Then we can pass the topology on \( A \) to \( \lim A/A_i \) making the latter into a topological group isomorphic to \( A \). Let \( \{a_k\} \) be a Cauchy sequence. Then for all \( i \) there is an \( N_i \) such that \( \forall k,l \geq N_i, a_k \equiv a_l \pmod{A_i} \). It follows that \( \langle \varphi(a_k) \rangle \) converges in \( \lim A/A_i \) and so \( a_k \) must converge in \( A \), implying \( A \) is complete. Conversely, suppose \( A \) is complete. Take any \( a \in A, a \neq 0 \). To show that \( \varphi \) is injective, it suffices to show that \( a \in A_i \) for some \( i \). This follows directly from the Hausdorff property of \( A \). Since \( a \neq 0 \) and for all \( i, A_i \) is an open set containing 0, there must be some \( i \) for which \( a \not\in A_i \). Now we prove that \( \varphi \) is surjective. For this, we need an explicit description for \( \lim A/A_i \). We take the one given in Weibel’s text, which is \( \lim A/A_i = \ker(\Delta) \), where

\[
\prod_{i=0}^{\infty} A/A_i \overset{\Delta}{\longrightarrow} \prod_{i=0}^{\infty} A/A_i
\]

maps

\[
(\cdots, a_i, \cdots) \to (\cdots, \bar{a}_i + f_{i+1}(\bar{a}_{i+1}), \cdots, \bar{a}_1 - f_2(\bar{a}_2), a_0 - f_1(a_1)).
\]

Here the overline notation denotes the image under the appropriate quotient, and take \( f_i : A_i \to A_{i-1} \) to be the given inclusions. The map \( \varphi : A \to \lim A/A_i \) is then just the appropriate quotient map in each factor. Now take any element \( (\cdots, \bar{a}_i, \cdots, \bar{a}_1, 0) \in \ker(\Delta) \) and choose representatives \( a_k \) of \( \bar{a}_k \) for each \( k \). We note that the zeroth coordinate is determined to be 0 since \( A/A_0 = A/A = 0 \). We then have that

\[
\bar{a}_i = f_{i+1}(\bar{a}_{i+1}), \quad \forall i > 0
\]

\[
\Rightarrow a_i \equiv a_{i+1} \pmod{A_i}, \quad \forall i > 0.
\]

However since each \( A_i \) is included in \( A_{j} \) for \( j < i \), we also have that

\[
a_j \equiv a_{i+1} \pmod{A_j}, \quad \forall 0 < j \leq i.
\]

It follows that \( \{a_i\} \) is in fact a Cauchy sequence in \( A \), so by our completeness assumption, it must converge to some \( a \in A \). Moreover, for all \( i > 0 \) we have for all \( k \geq i \),

\[
a_k - a_i \in A_i \Rightarrow a_k \equiv a_i \pmod{A_i} \Rightarrow a \equiv a_i \pmod{A_i}.
\]

Thus,

\[
\varphi(a) = (\cdots, \bar{a}_i, \cdots, \bar{a}_1, 0),
\]

implying \( \varphi \) is surjective, hence an isomorphism.
**Solution 38. Exercise 38**

Suppose \( \{A_j\} \) is a tower of finite abelian groups. Since each \( A_j \to A_k \) factors through \( A_j \to A_{j+1} \), we have that

\[
\text{im}(A_j \to A_k) \subseteq \text{im}(A_{j-1} \to A_k),
\]

for all \( j > k \). Thus, for each \( k \), \( \text{im}(A_j \to A_k) \) is a descending chain of subgroups of \( A_k \). Since \( A_k \) is finite, this chain must stabilize, that is, there must exist \( j \) such that \( \text{im}(A_i \to A_k) = \text{im}(A_j \to A_k) \) for all \( i \geq j \). Thus, \( \{A_i\} \) satisfies the Mittag-Leffler condition, implying \( \lim^1 A_i = 0 \).

Similarly, suppose \( \{A_i\} \) is a tower of finite-dimensional vector spaces over a field. Since each \( A_j \to A_k \) factors through \( A_j \to A_{j-1} \),

\[
\text{im}(A_j \to A_k) \subseteq \text{im}(A_{j-1} \to A_k),
\]

for all \( j > k \). It follows that

\[
\dim(\text{im}(A_j \to A_k)) \leq \dim(\text{im}(A_{j-1} \to A_k)),
\]

for all \( j > k \). Since \( \dim A_k \) is finite for all \( k \) this is a descending chain of integers which is bounded below by 0 and above by \( \dim A_k \), hence must stabilize. It follows that the dimension, hence image must stabilize for large enough \( j \).

**Solution 39. Exercise 39**

Our first task is to show that \( \text{Ext}^1_Z(\mathbb{Z}[p^{-1}], \mathbb{Z}) \cong \mathbb{Z}/p \mathbb{Z} \) using that \( \mathbb{Z}[p^{-1}] = \bigcup_i p^{-i} \mathbb{Z} \). We have the following short exact sequence

\[
0 \longrightarrow \lim^1 \text{Hom}(p^{-i} \mathbb{Z}, \mathbb{Z}) \longrightarrow \text{Ext}^1_Z(\mathbb{Z}[p^{-1}], \mathbb{Z}) \longrightarrow \lim \text{Ext}^2_Z(p^{-i} \mathbb{Z}, \mathbb{Z}) \longrightarrow 0
\]

Since \( p^{-i} \mathbb{Z} \cong \mathbb{Z} \) for all \( i \), \( \text{Ext}^1_Z(p^{-i} \mathbb{Z}, \mathbb{Z}) = 0 \) for all \( i \) implying the rightmost term above is zero. Thus we have an isomorphism \( \lim^1 \text{Hom}(p^{-i} \mathbb{Z}, \mathbb{Z}) \cong \text{Ext}^1_Z(\mathbb{Z}[p^{-1}], \mathbb{Z}) \). Now, we have the sequence

\[
\mathbb{Z} \to p^{-1} \mathbb{Z} \to p^{-2} \mathbb{Z} \to \cdots
\]

which we apply the Hom functor to get

\[
\text{Hom}(\mathbb{Z}, \mathbb{Z}) \leftarrow \text{Hom}(p^{-1} \mathbb{Z}, \mathbb{Z}) \leftarrow \text{Hom}(p^{-2} \mathbb{Z}, \mathbb{Z}) \leftarrow \cdots.
\]

Note that since \( p^{-i} \mathbb{Z} \cong \mathbb{Z} \) each of the terms above is isomorphic to \( \mathbb{Z} \) via the isomorphism \( f \mapsto f(1) \). However, since \( p^i f(p^{-i}) = f(1) \), we have that for each \( i \), \( f(1) \) must be a multiple of \( p^i \). So we can interpret the map \( f \mapsto f(1) \) instead as an isomorphism \( \text{Hom}(p^{-1} \mathbb{Z}, \mathbb{Z}) \cong p^{-1} \mathbb{Z} \). The sequence of Hom-groups above is then nothing but

\[
\mathbb{Z} \leftarrow p \mathbb{Z} \leftarrow p^2 \mathbb{Z} \leftarrow \cdots,
\]

where the maps are all multiplication by \( p \). It follows that \( \lim^1 \text{Hom}(p^{-i} \mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p \mathbb{Z} \), as desired.

**Solution 40. Exercise 40**

Let \( \begin{array}{c} y \\ \downarrow \psi \\ x \end{array} \rightarrow \begin{array}{c} \phi \\ \downarrow \psi \\ z \end{array} \) denote the poset displayed below

\[
\begin{array}{ccc}
 & y & \\
 & \downarrow \psi & \\
x & \phi \\
\downarrow \psi & \\
z
\end{array}
\]
Let $A$ be a $\cdots \longrightarrow$-shaped diagram of $R$-modules. Following the construction in Weibel, we let

$$C_0 = A_x \times A_y \times A_z, \quad C_1 = A_x^{(x)} \times A_z^{(y)},$$

and define two differentials between them as follows. Let $d^0 : C_0 \to C_1$ be given by the matrix

$$d^0 = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \psi & 0 \end{pmatrix},$$

and $d^1 : C_0 \to C_1$ be given by

$$d^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the derived functors of the lim functor of $A$ are the cohomology of the chain complex

$$0 \to C_0 \xrightarrow{d^0-d^1} C_1 \to 0.$$

Evidently, the zeroth cohomology is given by

$$\ker(d^0-d^1) = \{(a,b,c) | \varphi(a) = \psi(b) = c \} \subset A_x \times A_y \times A_z \implies H^0(C) \cong \{(a,b) | \varphi(a) = \psi(b) \} \subset A_x \times A_y,$$

which is the pullback, as expected. The lim$^1$ functor in this case is the first cohomology of this cochain complex, hence is the cokernel $\text{coker}(d^0-d^1) = H^1(C)$. To see that this is the same as the cokernel of the difference map $A_x \times A_y \to A_z$ mapping $(a,b) \mapsto \varphi(a) - \psi(b)$, we first note that the diagonal $\Delta = \{(c,c) | c \in A_z \} \subset A_x \times A_z$ is contained in the image of $d^0 - d^1$. It follows that the map $A_x \times A_y \to H^1(C)$ factors through the map $A_x \times A_y \xrightarrow{\varphi-\psi} A_z$ given by $q(c,d) = c - d$. That these cokernels do indeed coincide then follows from commutativity of the following diagram with exact rows.

$$
\begin{array}{ccc}
A_x \times A_y \times A_z & \xrightarrow{d^0-d^1} & A_x^{(x)} \times A_z^{(y)} \\
\downarrow & & \downarrow q \\
A_x \times A_y & \xrightarrow{\varphi-\psi} & A_z \\
\end{array}
\xrightarrow{\text{coker}(\varphi-\psi)} 0
$$

To see this, note that since the quotient $A_x^{(x)} \times A_z^{(y)} \to H^1(C)$ factors through $q : A_x^{(x)} \times A_z^{(y)} \to A_z$, we have that $H^1(C)$ is also the cokernel of the composition

$$A_x \times A_y \times A_z \xrightarrow{\varphi-\psi} A_x \times A_y \xrightarrow{\varphi-\psi} A_z.$$

However, since the projection $A_x \times A_y \times A_z \to A_x \times A_y$ is surjective, this is the same as $\text{coker}(\varphi-\psi)$.

**Solution 41. Exercise 41**

Going off of Example 5.2.6 in Weibel, since $E^r_{pq}$ is a spectral sequence converging to $H_*$, for each $n$ we have a filtration

$$0 = F_{-1}H_n \subset F_0H_n \subset \cdots \subset F_pH_n \subset \cdots F_nH_n = H_n$$

such that

$$E^\infty_{pq} \cong \frac{F_pH_{p+q}}{F_{p-1}H_{p+q}}.$$

In this particular case, the only nonzero columns of the $E^2$-page are $p = 0, 1$. Since $d^2 : E^2_{pq} \to E^2_{p-2,q+1}$, and for $p = 0, 1$ we have $p - 1 = -2, -1$, it follows that all differentials on this $E^2$-page are zero and thus, $E^2 = E^3 = \cdots = E^\infty$. We then have

$$E^2_{0n} = \frac{F_0H_n}{F_{-1}H_n} = F_0H_n \quad \text{and} \quad E^2_{1,n-1} = \frac{F_1H_n}{F_0H_n},$$

and

$$E^\infty_{0n} \equiv \frac{F_0H_n}{F_{-1}H_n} = F_0H_n \quad \text{and} \quad E^\infty_{1,n-1} \equiv \frac{F_1H_n}{F_0H_n}.$$
implying there is a short exact sequence
\[ 0 \to E^2_{0,n} \to F_1H_n \to E^2_{1,n-1} \to 0. \]

However, for any \( k > 1 \) we have
\[ \frac{F_k H_n}{F_{k-1} H_n} \cong E^2_{k,n-k} = 0 \implies F_k H_n = F_{k-1} H_n. \]

Thus, \( H_n = F_n H_n \cong F_1 H_n \) and we get the desired short exact sequence below
\[ 0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0. \]

**Solution 42. Exercise 42**

In this exercise, \( E^* \) is a spectral sequence converging to \( H_* \) such that \( E^r_{pq} = 0 \) for \( q \neq 0, 1 \). As in the preceding exercise, for each \( n \), we have a filtration
\[ 0 = F_{-1} H_n < F_0 H_n < \cdots < F_p H_n < \cdots F_n H_n = H_n, \]
satisfying
\[ E^\infty_{pq} \cong \frac{F_p H_{p+q}}{F_{p-1} H_{p+q}}. \]

In this case, however, we do not have the luxury of every differential being zero, since \( d^2 : E^2_{p0} \to E^2_{p-2,1} \) is not necessarily zero for \( p > 1 \). Still, since \( d^3 : E^3_{pq} \to E^3_{p-3,q+2} \) and the \( E^3 \)-page can have no more than the two rows \( q = 0, 1 \), we have that all the differentials on this third page are zero, hence \( E^3 \to E^4 = \cdots = E^\infty \). Thus
\[ \ker(d^2_p) \cong E^\infty_{p0} \cong \frac{F_p H_p}{F_{p-1} H_p} = \frac{H_p}{F_{p-1} H_p}, \forall p \geq 2 \quad \text{and} \quad \frac{E^2_{p1}}{\operatorname{im}(d^2_{p+2})} \cong E^\infty_{p1} \cong \frac{F_p H_{p+1}}{F_{p-1} H_{p+1}}, \forall p \geq 0. \]

From the isomorphism on the left above, we have that the following sequence is exact for all \( p \geq 2 \),
\[ 0 \to F_{p-1} H_p \to H_p \to E^2_{p0} \xrightarrow{d} E^2_{p-2,1}. \]

Now note that
\[ \frac{F_{p-1} H_{p+1}}{F_{p-2} H_{p+1}} \cong E^\infty_{p-1,2} = 0 \implies F_{p-1} H_{p+1} = F_{p-2} H_{p+1}, \forall p \geq 0. \]

Thus
\[ F_{p-1} H_{p+1} = F_0 H_{p+1} \cong E^\infty_{0,p+1} = 0, \forall p \geq 1. \]

Using the isomorphism to the right above, we can deduce that
\[ \frac{E^2_{p1}}{\operatorname{im}(d^2_{p+2})} \cong F_p H_{p+1}, \forall p \geq 1. \]

Thus, we can extend the exact sequence obtained above to the long exact sequence below
\[ \cdots \to E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p0} \to E^2_{p-2,1} \to \cdots. \]

Evidently, this will go on to the left indefinitely. On the right we can use the fact that \( d^2_0 = d^2_1 = 0 \) to get
\[ E^2_{00} = E^\infty_{00} \cong \frac{F_0 H_0}{F_{-1} H_0} = F_0 H_0 = H_0 \quad \text{and} \quad \frac{E^2_{10}}{F_0 H_1} = \frac{E^\infty_{10}}{F_0 H_1} = \frac{F_1 H_1}{F_0 H_1}. \]
Now, using the isomorphism on the right above in the case \( p = 0 \), we have

\[
\frac{E_{p1}^2}{\text{im}(d_{p2}^2)} \cong F_0H_1,
\]

implying that the following sequence is exact

\[
H_2 \to E_{20}^2 \to E_{01}^2 \to H_1 \to E_{10}^2 \to 0 \to H_0 \cong E_{00}^2 \to 0.
\]

The sequence can then be continued as above for \( p \geq 2 \), so we are done.

**Solution 43. Exercise 43**

In this exercise, \( E_{**} \) is a double complex whose only nonzero columns are at \( p \) and \( p - 1 \), \( E_{pq}^1 = H_q(E_{p*}) \) is the vertical homology at site \( (p,q) \) and \( E_{pq}^2 = H_p(E_{*q}) \) is the horizontal homology of the resulting sequence of complexes \( E_{**}^1 \). Our goal is to show that there is a short exact sequence

\[
0 \to E_{p-1,q+1}^2 \to H_{p+q}(T) \to E_{pq}^2 \to 0,
\]

where \( T \) denotes the total complex of \( E_{**} \). The point here is that computing the vertical and horizontal homology consecutively is, in a sense, an approximation to the homology of the total complex, at least when there are only two nonzero columns. Here we take the straightforward approach and compute each of these groups explicitly. After stumbling through some definitions, one finds

\[
H_{p+q}(T) = \frac{\{(a,b) \in E_{p-1,q+1} \times E_{pq} | d^v(a) + d^h(b) = d^v(b) = 0\}}{\{(d^v(x) + d^h(y), d^v(y))(x,y) \in E_{p-1,q+2} \times E_{p,q+1}\}},
\]

\[
E_{pq}^2 = \ker(\tilde{d}^h : H_q(E_{p*}) \to H_q(E_{p-1,*})) = \frac{\{b \in E_{pq} | d^v(b) = 0, d^h(b) = d^v(x), x \in E_{p-1,q+1}\}}{\{d^v(y) | y \in E_{p,q+1}\}},
\]

and

\[
E_{p-1,q+1}^2 = \frac{H_{q+1}(E_{p-1,*})}{d^h(H_q(E_{p*}))} = \frac{\{a \in E_{p-1,q+1} | d^v(a) = 0\}}{\{d^v(x) | x \in E_{p-1,q+2}\} + \{d^h(y) | y \in E_{p,q+1}\}}.
\]

From this we see automatically that there are maps

\[
H_{p+q}(T) \xrightarrow{\alpha} E_{pq}^2, \quad \alpha((a,b)) = [b],
\]

and

\[
E_{p-1,q+1}^2 \xrightarrow{\beta} H_{p+q}(T), \quad \beta([a]) = [a,0].
\]

Evidently \( \alpha \) is surjective, \( \beta \) is injective, and \( \text{im}(\beta) \subseteq \ker(\alpha) \). For the reverse inclusion, note that if \( [a,b] \in \ker(\alpha) \) then \( d^v(y) = b \) for some \( y \in E_{p,q+1} \). Thus \( [a,b] = [a,0] + [0,d^v(y)] \implies [a,b] = [a,0] \in \ker(\alpha) \), implying that the sequence

\[
0 \to E_{p-1,q+1}^2 \to H_{p+q}(T) \to E_{pq}^2 \to 0
\]

is exact.

**Solution 44. Exercise 44**

**Solution 45. Exercise 45**
Solution 46. Exercise 46

The task that this exercise bestows upon us is to prove the Universal Coefficient Theorem for cohomology using a spectral sequence argument. The theorem states that if \((P_, d_\cdot)\) is a chain complex of projective \(R\)-modules such that \(d_n(P_n)\) is also projective, for all \(n\), then for any other \(R\)-module, \(M\), there is a split exact sequence

\[
0 \to \text{Ext}^1_R(H_{n-1}(P_), M) \to H^n(\text{Hom}_R(P,, M)) \to \text{Hom}_R(H_n(P_), M) \to 0.
\]

In the following we will suppress the subscript \(R\). Let \(M \to I_\cdot\) be an injective resolution, and consider the following double (co)chain complex.

\[
\begin{array}{ccccccc}
0 & \to & \text{Hom}(P_1, I_0) & \to & \text{Hom}(P_1, I_1) & \to & \text{Hom}(P_1, I_2) & \to & \cdots \\
& & \uparrow d_2 & & \uparrow d_2 & & \uparrow d_2 & & \\
0 & \to & \text{Hom}(P_0, I_0) & \to & \text{Hom}(P_0, I_1) & \to & \text{Hom}(P_0, I_2) & \to & \cdots \\
& & \uparrow d_1 & & \uparrow d_1 & & \uparrow d_1 & & \\
0 & \to & \text{Hom}(P_{-1}, I_0) & \to & \text{Hom}(P_{-1}, I_1) & \to & \text{Hom}(P_{-1}, I_2) & \to & \cdots \\
& & \uparrow d_0 & & \uparrow d_0 & & \uparrow d_0 & & \\
\end{array}
\]

We first show that the spectral sequence for a double complex is in fact computing what we want, namely \(H^n(\text{Hom}(P_, M))\). To do this, we filter the complex vertically and run our spectral sequence. In this case, the II\(E_0\) is given by:

\[
\begin{array}{ccccccc}
IIE_0: & & & & & & \\
0 & \to & \text{Hom}(P_1, I_0) & \to & \text{Hom}(P_1, I_1) & \to & \text{Hom}(P_1, I_2) & \to & \cdots \\
0 & \to & \text{Hom}(P_0, I_0) & \to & \text{Hom}(P_0, I_1) & \to & \text{Hom}(P_0, I_2) & \to & \cdots \\
0 & \to & \text{Hom}(P_{-1}, I_0) & \to & \text{Hom}(P_{-1}, I_1) & \to & \text{Hom}(P_{-1}, I_2) & \to & \cdots \\
\end{array}
\]

Since injective resolutions are exact sequences, and \(\text{Hom}(P_k, _)\) is exact by virtue of \(P_k\) being projective, the homology of these rows vanishes above degree zero. We then obtain the following for II\(E_1\).
Evidently, the cohomology of this cochain complex is precisely what we want to find. Furthermore, once this cohomology is computed to obtain the $E^2$ page, since there is only a single nonzero column, the differentials $d_r$ for $r \geq 2$ must vanish. Thus $\overset{\infty}{II}E^p_q = \overset{2}{II}E^p_q = H^n(\text{Hom}(P, M))$.

Now we filter this complex horizontally and run the spectral sequence in hopes of gaining a different description of this cohomology. The $E_0$-page is then given by:

Now since each $I_n$ is injective, the functor $\text{Hom}(\_, I_n)$ is exact and so it commutes with homology, i.e. $H^n(\text{Hom}(P, I_n)) = \text{Hom}(H_n(P), I_n)$. The $E_1$-page is then shown below.
It follows that $E_2^{pq} = \text{Ext}^p(H_q(P), M)$. However, note that since $d_n(P_n)$ is also projective for all $n$, $Z_n = \text{ker}(d_n)$ must also be projective. To see this, it suffices to notice that the exact sequence
\[ 0 \to Z_n \to P_n \to d_n(P_n) \to 0 \]
 splits since $d_n(P_n)$ is projective. Thus $Z_n$ is a direct summand of $P_n$, a projective module, hence must be projective. It follows that
\[ 0 \to d_n(P_n) \to Z_n \to H_n(P) \to 0 \]
is a projective resolution for $H_n(P)$ for all $n$. Then, computing $\text{Ext}^q(H_p(P), M)$ using this resolution for $H_p(P)$ shows that $\text{Ext}^p(H_q(P), M) = 0$ for $p \geq 2$. This leads us to the following description of the $E_2$-page shown below.

\[
\begin{array}{cccc}
0 & \text{Hom}(H_1(P), M) & \text{Ext}^1(H_1(P), M) & 0 \\
0 & \text{Hom}(H_0(P), M) & \text{Ext}^1(H_0(P), M) & 0 \\
0 & \text{Hom}(H_{-1}(P), M) & \text{Ext}^1(H_{-1}(P), M) & 0 \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

It now follows, from an argument dual to that of Exercise 41, that for all $n$ there is a short exact sequence
\[ 0 \to \text{Ext}^1(H_{n-1}(P), M) \to H^n(\text{Hom}(P, M)) \to \text{Hom}(H_n(P), M) \to 0. \]

It remains to show that this sequence is in fact split.

**Solution 47. Exercise 47**

Let $L \in \text{mod-R}$, $M \in \text{R-mod}$, $M \in \text{mod-S}$, and $N \in \text{S-mod}$. Choose projective resolutions $P. \to L$ and $Q. \to N$. We are lead to consider the double complex below.

\[
\begin{array}{cccc}
0 & P_0 \otimes M \otimes Q_2 & P_1 \otimes M \otimes Q_2 & P_2 \otimes M \otimes Q_2 & \cdots \\
0 & P_0 \otimes M \otimes Q_1 & P_1 \otimes M \otimes Q_1 & P_2 \otimes M \otimes Q_1 & \cdots \\
0 & P_0 \otimes M \otimes Q_0 & P_1 \otimes M \otimes Q_0 & P_2 \otimes M \otimes Q_0 & \cdots \\
0 & 0 & 0 & 0 & \\
\end{array}
\]
Running our first spectral sequence on this double complex gives us the following $^1E^0$-page.

$$
\begin{array}{cccc}
  & \vdots & \vdots & \vdots \\
 0 & P_0 \otimes M \otimes Q_2 & P_1 \otimes M \otimes Q_2 & P_2 \otimes M \otimes Q_2 \\
 0 & P_0 \otimes M \otimes Q_1 & P_1 \otimes M \otimes Q_1 & P_2 \otimes M \otimes Q_1 \\
 0 & P_0 \otimes M \otimes Q_0 & P_1 \otimes M \otimes Q_0 & P_2 \otimes M \otimes Q_0 \\
 0 & 0 & 0 & 0 \\
\end{array}
$$

Taking homology gives us the next page displayed below.

$$
\begin{array}{cccc}
  & \vdots \\
 0 & P_0 \otimes \text{Tor}^S_2(M,N) & P_1 \otimes \text{Tor}^S_2(M,N) & P_2 \otimes \text{Tor}^S_2(M,N) \\
 0 & P_0 \otimes \text{Tor}^S_1(M,N) & P_1 \otimes \text{Tor}^S_1(M,N) & P_2 \otimes \text{Tor}^S_1(M,N) \\
 0 & P_0 \otimes \text{Tor}^S_0(M,N) & P_1 \otimes \text{Tor}^S_0(M,N) & P_2 \otimes \text{Tor}^S_0(M,N) \\
 0 & 0 & 0 & 0 \\
\end{array}
$$

Notice that we used the fact that since $P_p$ is projective for all $p$, $P_p \otimes -$ is exact, hence commutes with homology so that $\text{Tor}^S_q(P_p \otimes M,N) = P_p \otimes \text{Tor}^S_q(M,N)$. Taking homology then gives us that

$$
^1E^2_{pq} = \text{Tor}^R_p(L,\text{Tor}^S_q(M,N)).
$$

A symmetric argument shows that if we run our spectral sequence with the opposite filtration we get

$$
^1E^2_{pq} = \text{Tor}^S_p(\text{Tor}^R_q(L,M),N).
$$

Now note that if $M$ is a flat $S$-module, then

$$
^1E^2_{pq} = \begin{cases} 
  \text{Tor}^R_p(L,M \otimes_S N) & q = 0 \\
  0 & q \neq 0 
\end{cases}
$$

Since this consists of a single row, this spectral sequence collapses at the $E^2$-page and so it converges to $\text{Tor}^R(L,M \otimes_S N)$. Since we started with a first quadrant spectral sequence, both of these spectral sequences converge to the same thing, namely $H_*(\text{Tot}(P \otimes M \otimes Q))$. Thus, in this case, we have

$$
^1E^2_{pq} = \text{Tor}^S_p(\text{Tor}^R_q(L,M),N) \Rightarrow \text{Tor}^R(L,M \otimes_S N).
$$
Similarly, if $M$ is a flat $R$-module, then

\[ E^2_{pq} = \begin{cases} \text{Tor}_p^S(L \otimes_R M, N), & q = 0 \\ 0 & q \neq 0 \end{cases} \]

Again, since there is only one nonzero row, this spectral sequence collapses at $E^2$ and so it converges to $\text{Tor}_p^S(L \otimes_R M, N)$. Since the two converge to the same object, we have

\[ I^E_{pq} = \text{Tor}_p^S(\text{Tor}_q^R(L, M), N) \implies \text{Tor}_p^S(L \otimes_R M, N). \]

**Solution 48. Exercise 49**

Suppose, for a contradiction, that there is a map $w : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}[-1]$ that makes the sequence

\[ \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{w} \mathbb{Z}/2\mathbb{Z}[-1] \]

into an exact triangle. We then have a solid-arrow diagram below in which the square on the left commutes.

\[
\begin{array}{c}
\mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{w} \mathbb{Z}/2\mathbb{Z}[-1] \\
\downarrow 1 \quad \downarrow 1 \quad \downarrow 1 \quad \downarrow 1 \\
\mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{v} \text{cone}(2) \xrightarrow{\delta} \mathbb{Z}/2\mathbb{Z}[-1]
\end{array}
\]

The bottom row is, by definition, an exact triangle. Thus, by axiom TR3, there must exist a map $h$ shown above that makes the diagram commute. However, the degree zero part of the center square looks like

\[
\begin{array}{c}
\mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \\
\downarrow 1 \quad \downarrow 1 \\
\mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/4\mathbb{Z}
\end{array}
\]

We have now reached a contradiction since the identity $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ does not factor through $\mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z}$. Thus, the sequence on top cannot be an exact triangle.

**Solution 49. Exercise 50**

We are given an exact triangle $(u, v, w)$ on $(A, B, C)$ and want to show that $vu = vw = (Tu)w = 0$. We know that $(1_A, 0, 0)$ is an exact triangle on $(A, A, 0)$, so by axiom TR3, there is a dashed arrow below making the diagram commute. However, since the source of this arrow is 0, this map can only be the zero map. Thus, by commutativity of the middle square below, $vu = 0$.

\[
\begin{array}{c}
A \xrightarrow{1_A} A \xrightarrow{0} TA \\
\downarrow 1_A \quad \downarrow u \quad \downarrow \exists h = 0 \quad \downarrow T1 \\
A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA
\end{array}
\]

Now apply the same reasoning to the rotated triangle $(v, w, -Tu)$ to obtain $wv = 0$. Then since $w$ and $-Tu$ are the $v$ and $w$ of this rotated triangle, $(-Tu)w = 0 \implies (Tu)w = 0$.

**Solution 50. Exercise 51**

For this problem, we are asked to prove a form of the five-lemma for triangulated categories. Specifically, given a morphism of exact triangles shown below, in which $f$ and $g$ are isomorphisms, we want to show that $h$ must also be an isomorphism.
One is tempted to simply extend this diagram to the right with $Tg : TB \to TB'$ and use the standard five-lemma, however, this fails because our category is not necessarily abelian. Our approach will be to carry this diagram into an abelian category and apply the five-lemma there to obtain our result. First, we prove an auxiliary result.

**Lemma.** Let $f : X \to Y$ be a morphism in $K$. If $f^* = \text{Hom}_K(C, f)$ is an isomorphism for all $C \in \text{Ob}(K)$, then so is $f$.

**Proof.** We view $f^*$ as a natural transformation $h_X = \text{Hom}_K(_, X) \to \text{Hom}_K(_, Y) = h_Y$. The Yoneda Lemma says that there is a bijection

$$\text{Hom}_K(h_X, h_Y) \approx \text{Hom}_K(X, Y),$$

which preserves composition. Thus, natural isomorphisms correspond to isomorphisms and the result follows. \[\square\]

Now for any $W \in \text{Ob}(K)$, applying $\text{Hom}_K(W, _) \to$ to the diagram above yields a similar diagram in the category of abelian groups. This is because $K$ is additive, being triangulated, hence

$$\text{Hom}_K(W, _) : K \to \text{Ab}$$

for all $W$. The five lemma then applies to show that $\text{Hom}_K(W, h)$ is an isomorphism for all $W$. The result then follows from the lemma above.

**Solution 51. Exercise 52**

Consider the morphism of chain complexes shown below.

$$
\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/4 \longrightarrow \cdots 
$$

This is not nullhomotopic since finding a nullhomotopy amounts to finding a lift shown below, which evidently does not exist.

$$
\begin{array}{c}
\mathbb{Z}/2 \\
\downarrow 2 \\
\mathbb{Z}/4
\end{array} \quad \begin{array}{c}
\mathbb{Z}/4 \\
\downarrow 2 \\
\mathbb{Z}/4
\end{array}
$$

Still, the chain complex below is evidently exact and so it is quasi-isomorphic to the zero complex. Composing this morphism of chain complexes with this quasi-isomorphism shows that this morphism is in fact zero in the derived category.