Let X be a topological space containing a point x that does not have a smallest open neighborhood. Then the category $\mathrm{Ab}(X)$ of sheaves of abelian groups on X does not have enough injectives.

1. Suppose that U and V are open subsets of X. Show that the map $\mathbf{Z}_U \oplus \mathbf{Z}_V \to \mathbf{Z}_X$ is surjective if and only if $U \cup V = X$. Prove as well that, for any sheaf F, the map $F_U \oplus F_V \to F$ is surjective if $U \cup V = X$. (What condition on F guarantees the converse?)

2. Suppose that there is a projective sheaf P and a map $P \to \mathbf{Z}_X$. Prove that for any open neighborhood U of x in X, the map $P(U) \to \mathbf{Z}_X(U) = \mathbf{Z}$ is the zero map. (Hint: Consider the cover $(X - \{x\}) \cup V = X$ where $V \subsetneq U$ is a smaller open neighborhood of x in X.)

3. Conclude that if P is projective and x is a point of X with no smallest open neighborhood then $P_x = 0$.