In class, we defined an abelian category to be a category A in which the following conditions hold:

1. Finite products and finite coproducts exist, and the map

$$\prod_{i\in I} X_i \to \prod_{i\in I} X_i$$

is an isomorphism.

- 2. Finite limits and finite colimits exist.
- 3. Coimages and images coincide.

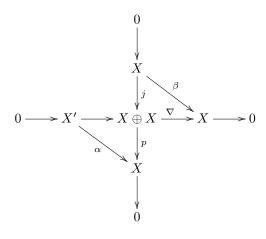
Show that this definition implies Grothendieck's using the following steps:

- 1. Show that the empty product is a final object and the empty coproduct is an initial object. Conclude that A has a zero object, denoted 0.
- 2. Show that kernels can be constructed as finite limits and cokernels can be constructed as finite colimits (both involving the zero object).
- 3. If  $f, g: X \to Y$  are maps in A, define f + g as the composition:

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \oplus Y \xrightarrow{\nabla} Y$$

Construct the middle arrow using universal properties. ( $\Delta$  is the diagonal,  $\nabla$  the sum.)

- 4. Show that the addition law f + g defined above maked Hom(X, Y) into a commutative monoid with unit the zero homomorphism  $X \to 0 \to Y$ .
- 5. Let  $X' = \ker(\nabla : X \oplus X \to X)$ . Let  $p, q : X \oplus X \to X$  be the first projection. Prove that  $X' \to X \oplus X \xrightarrow{p} X$  is an isomorphism. Define  $-\operatorname{id}_X = q \circ p^{-1}$ . Hint: construct a commutative diagram



where  $j:X\to X\oplus X$  is 0 on the first factor and  $\mathrm{id}_X$  on the second. First show  $\beta=\mathrm{id}_X,$  then

6. Show that  $\operatorname{Hom}(X, Y)$  is an abelian group (define  $-f = f \circ (-\operatorname{id}_X)$ ).