In class, we defined an abelian category to be a category A in which the following conditions hold:

1. Finite products and finite coproducts exist, and the map
$$\coprod_{i \in I} X_i \to \prod_{i \in I} X_i$$
is an isomorphism.

2. Finite limits and finite colimits exist.

3. Coimages and images coincide.

Show that this definition implies Grothendieck's using the following steps:

1. Show that the empty product is a final object and the empty coproduct is an initial object. Conclude that A has a zero object, denoted 0.

2. Show that kernels can be constructed as finite limits and cokernels can be constructed as finite colimits (both involving the zero object).

3. If $f, g : X \to Y$ are maps in A, define $f + g$ as the composition:
$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \oplus Y \xrightarrow{\nabla} Y$$
Construct the middle arrow using universal properties. (Δ is the diagonal, ∇ the sum.)

4. Show that the addition law $f + g$ defined above maked $\mathrm{Hom}(X, Y)$ into a commutative monoid with unit the zero homomorphism $X \to 0 \to Y$.

5. Let $X' = \ker(\nabla : X \oplus X \to X)$. Let $p, q : X \oplus X \to X$ be the first projection. Prove that $X' \to X \oplus X \xrightarrow{p} X$ is an isomorphism. Define $-\mathrm{id}_X = q \circ p^{-1}$. Hint: construct a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & X \arrow[d, "j"] \arrow[dr, "\beta"] & & \\
0 \arrow[r] & X' \arrow[r] \arrow[dr, "\alpha"'] & X \oplus X \arrow[r, "\nabla"] \arrow[d, "p"] & X \arrow[r] & 0 \\
 & & X \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where $j : X \to X \oplus X$ is 0 on the first factor and id_X on the second. First show $\beta = \mathrm{id}_X$, then

6. Show that $\mathrm{Hom}(X, Y)$ is an abelian group (define $-f = f \circ (-\mathrm{id}_X)$).