Suppose I is a category. A *cospan* from $i \in I$ to $j \in I$ is a diagram $i \leftarrow k \rightarrow j$. The cospans from i to j form a category, in which a morphism is a commutative diagram:

$$\begin{tikzcd}
 & k \arrow[dd] & \\
i \arrow[ur] \arrow[dr] & & j \arrow[ul] \arrow[dl] \\
 & k' &
\end{tikzcd}$$

A diagram I is called *cosifted* if, for all i and j in I, the category of cospans from i to j is connected.

1. Prove that, if I is cosifted and $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ are diagrams of sets then $\varinjlim(X_i \times Y_i) \to \varinjlim(X_i) \times \varinjlim(Y_i)$ is a bijection.

2. Give an example of a noncosifted colimit and finite product that do not commute.

3. Let $\mathscr{A}$ be an abelian category and let $\mathbf{K}(\mathscr{A})$ be the category of chain complexes valued in $\mathscr{A}$. Show that for each $X_\bullet \in \mathbf{K}(\mathscr{A})$, the category of quasi-isomorphisms $X'_\bullet \to X_\bullet$ is cosifted.

4. Conclude that $\mathscr{D}(\mathscr{A})$ satisfies AB1 (it has a zero object, has finite products and coproducts, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : X \amalg Y \to X \times Y$ is an isomorphism for all X and Y) and that $\mathrm{Ch}(\mathscr{A}) \to \mathscr{D}(\mathscr{A})$ is additive.

5. Suppose that $\mathrm{Ch}(\mathscr{A}) \to \mathscr{C}$ is a functor that takes quasi-isomorphisms to isomorphisms. Show that if $f, g : X \to Y$ are chain homotopic maps in $\mathrm{Ch}(\mathscr{A})$ then their images in $\mathscr{C}$ are the same.