1. Let $C$ be an abelian category. A correspondence $f : X \dashrightarrow Y$ from $X$ to $Y$ in $C$ is a subobject $f \subset X \oplus Y$. The composition of $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ is the image of $f \times_Y g$ under the map $X \oplus Y \oplus Z \to X \oplus Z$. A correspondence is a map if the projection $f \subset X \oplus Y \to X$ is an isomorphism. The inverse $f^{-1} \subset Y \oplus X$ of $f$ is the correspondence induced by the isomorphism $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : X \oplus Y \simeq Y \oplus X$. For a subobject $Z \subset X$, we write $f(X)$ for the image of $S \times_X f \to Y$.

   (a) Show that if $f : X \dashrightarrow Y$ then there is a universal $g : X' \to X$ and a universal $h : Y \to Y'$ such that $hfg$ is a morphism. Verify that $X' = f^{-1}(Y)$ and $Y' = Y/f(0)$.

   (b) Verify that, if $f : X \to Y$ is a morphism, then above construction produces $f^{-1} : \mathrm{im}(f) \to \mathrm{coim}(f)$.

   (c) Show that $f \circ f^{-1} \circ f = f$ for any correspondence $f : X \dashrightarrow Y$.

2. Let $C_{\bullet,\bullet}$ be a double complex with differential $d = d_0 + d_1$ where $d_0 : C_{p,q} \to C_{p-1,q}$ and $d_1 : C_{p,q} \to C_{p,q-1}$. Give $\mathrm{Tot}^{\Pi}(C_{\bullet,\bullet})$ and $\mathrm{Tot}^{\oplus}(C_{\bullet,\bullet})$ the filtration where

$$F_q \,\mathrm{Tot}^{\Pi}(C_{\bullet,\bullet})_n = \prod_{\substack{j \geq p \\ i+j=n}} C_{i,j} F_q \,\mathrm{Tot}^{\oplus}(C_{\bullet,\bullet})_n = \bigoplus_{\substack{j \geq p \\ i+j=n}} C_{i,j}$$

   Verify the following:

$$\mathrm{gr}_q Z_{p+q}(\mathrm{Tot}^{\oplus} C_{\bullet,\bullet}) = d_0^{-1}(0) \cap d_1^{-1} d_0 d_1^{-1}(0) \cap d_1^{-1} d_0 d_1^{-1} d_0 d_1^{-1}(0) \cap \cdots$$
$$\mathrm{gr}_q B_{p+q}(\mathrm{Tot}^{\oplus} C_{\bullet,\bullet}) = d_0(C_{p-1,q}) + d_1 d_0^{-1}(0) + d_1 d_0^{-1} d_1 d_0^{-1}(0) + \cdots$$
$$\mathrm{gr}_q H_{p+q}(\mathrm{Tot}^{\oplus} C_{\bullet,\bullet}) = \frac{d_0^{-1}(0) \cap d_1^{-1} d_0 d_1^{-1}(0) \cap d_1^{-1} d_0 d_1^{-1} d_0 d_1^{-1}(0) \cap \cdots}{d_0(C_{p-1,q}) + d_1 d_0^{-1}(0) + d_1 d_0^{-1} d_1 d_0^{-1}(0) + \cdots}.$$

3. Let $C^0_{p,q} = C_{p,q}$ with differential $d_0$. Let $C^1_{p,q} = H_{p,q}(C^1_{\bullet,\bullet}, d_0)$. Show that $C^1_{p,q} = \frac{d_0^{-1}(0)}{d_0(C_{p-1,q})}$. Show that $d_1$ induces a differential $d_1 : C^1_{p,q} \to C^1_{p,q-1}$.

4. For each $r \geq 1$, set $d_{r+1} = d_r d_{r-1}^{-1} d_r$. Show that

$$d_r = (d_1 d_0^{-1})^{r-1} d_1$$

   (as a correspondence) for all $r$.

5. Define $C^{r+1}_{p,q} = H_{p,q}(C^r_{\bullet,\bullet}, d_r)$, inductively. Verify that $d_r$ is well-defined on $C^r_{p,q}$.

6. Conclude that the groups $C^r_{p,q}$ 'converge' to $\mathrm{gr}_q H_{p+q}(\mathrm{Tot}\, C_{\bullet,\bullet})$ as $r \to \infty$. Feel free to assume that, for each $n$, there are only finitely many $(p, q)$ with $p + q = n$ and $C_{p,q} \neq 0$.

7. Compute the Mayer–Vietoris spectral sequence for the open cover $\mathbf{CP}^2 = U_0 \cup U_1 \cup U_2$ where $U_i$ is the image of the open subset of $(x_0, x_1, x_2) \in \mathbf{C}^3$ where $x_i \neq 0$.