Math 6210: Lecture notes

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Week 3: Induced topologies

3.1 The (generalized) quotient topology

Let $f : X \to Y$ be a function and suppose that X is a topological space. We may induce a topology on Y by defining $U \subset Y$ to be open when $f^{-1}(U) \subset X$ is open. When f is surjective, this is called the **quotient topology**.

Exercise 1. (a) Show that the topology on Y is the **finest** one in which f is continuous.

(b) Show that if Z is a topological space and $g : Y \to Z$ is a function then gf is continuous if and only if g is.

More generally, if we had a collection of functions $f_i : X_i \to Y$ then we can induce a topology on Y in which $U \subset Y$ is called open if $f_i^{-1}(U)$ is open for all i. Then the preceding exercise has an analogue:

Exercise 2. (a) Show that the topology on Y is the finest in which f_i is continuous for every i.

(b) Show that if Z is a topological space and $g : Y \to Z$ is a function then g is continuous if and only if gf_i is continuous for every i.

3.2 The (generalized) subspace topology

Let $f : X \to Y$ be a function and suppose that Y comes with a topology. We can induce a topology on X by declaring $U \subset X$ is open if $U = f^{-1}(U)$ for some open $U \subset Y$. When f is injective, this is called the **subspace topology**.

Exercise 3. (a) Show that the topology on X is the **coarsest** in which f is continuous.

(b) Show that if W is a topological space and $h : W \to X$ is a funciton then fh is continuous if and only if h is.

As with the quotient topology, we have a generalization: Suppose that $f_i : X \to Y_i$ are functions and that each Y_i is a topological space. We give the X the coarsest topology so that each of the functions f_i is continuous.

Exercise 4. Show that the collection of all subsets $f_i^{-1}(U)$ where $U \subset X$ is open form a subbasis for the topology of X. In effect, the subsets

$$f_{i_1}^{-1}(U_1) \cap \cdots \cap f_{i_k}^{-1}(U_k)$$

where each $U_j \subset X_{i_j}$ is open, form a basis for the topology of X.

3.3 The product topology

Let I be an indexing set and suppose that for each $i \in I$ we have a topological space X_i. Let X be the set $\prod_{i \in I} X_i$, equipped with the coarsest topology that makes all of the projection maps $p_i : X \to X_i$ continuous. By Exercise 4, a subset of $U \subset X$ is open if and only if there is a finite collection $i_1, \ldots, i_k \in I$ and open subsets $U_j \subset X_{i_j}$ such that

$$U = p_1^{-1}(U_1) \cap \cdots \cap p_k^{-1}(U_k).$$

Exercise 5. Note that if J is a subset of I there is a projection $p_J : \prod_{i \in I} X_i \to \prod_{i \in J} X_i$. Verify that $U \subset X$ is open if and only if there is a finite subset $J \subset I$ and an open subset $V \subset \prod_{i \in J} X_i$ with $U = p_J^{-1}(V)$.

3.4 Example: S^1 and $\mathbf{R}/\mathbf{Z}$

Give $\mathbf{R}/\mathbf{Z}$ the quotient topology from $q : \mathbf{R} \to \mathbf{R}/\mathbf{Z}$ and give S^1 the subspace topology from $i : S^1 \to \mathbf{R}^2$. Consider the function $f : \mathbf{R} \to \mathbf{R}^2$ defined by

$$f(t) = \big(\cos(2\pi t), \sin(2\pi t)\big).$$

Note that f is well defined on $\mathbf{R}/\mathbf{Z}$ and takes values in S^1; that is, there is a function $g : \mathbf{R}/\mathbf{Z} \to S^1$ such that $igq = f$.

Lemma 3.1. g is continuous.

Proof. By the universal property of the quotient topology, f is continuous if and only if gq is continuous; by the universal property of the subspace topology, igq is continuous if and only if gq is continuous. □

It is well-known that g is a bijection, so we omit a proof of this fact.

Lemma 3.2. f is open.

Proof. It is sufficient to show that $f(U)$ is open for all U in a basis for $\mathbf{R}$. We take for this basis the collection of intervals (a, b) such that $k\frac{\pi}{2} < a < b < (k+2)\frac{\pi}{2}$ and $k \in \mathbf{Z}$. Then if $k \equiv 0 \pmod 4$,

$$f((a,b)) = \big\{(x,y) \in S^1 \big| \cos(2\pi b) < x < \cos(2\pi a) \text{ and } y > 0\big\}$$

is the intersection of the open subset $(\cos(2\pi b), \cos(2\pi a)) \times (0, \infty) \subset \mathbf{R}^2$ with S^1. Similarly, if $k = 1 \pmod 4$,

$$f((a,b)) = \big\{(x,y) \in S^1 \big| \sin(2\pi b) < x < \sin(2\pi a) \text{ and } x < 0\big\}$$

is the intersection of $(-\infty, 0) \times (\sin(2\pi b), \sin(2\pi a)) \subset \mathbf{R}^2$ with S^1 and is therefore open in S^1. Similar remarks apply for $k \equiv 2, 3 \pmod 4$. □

Lemma 3.3. g is open.

Proof. Let U be an open subset of $\mathbf{R}/\mathbf{Z}$. Then $g(U) = g(q(q^{-1}(U))) = f(q^{-1}(U))$. Since q is continuous, $q^{-1}(U)$ is open, and since f is open, $g(U) = f(q^{-1}(U))$ is open. □

Proposition 3.4. g is a homeomorphism.

Proof. It is well-known that g is a bijection; since it is also open it is a homeomorphism. □

Week 4: The product topology and Tychonoff's theorem

4.1 The axioms of choice

The axiom of choice has many equivalent formulations. In my opinion, the most elegant is the first one below. Before stating it, let us recall that a **section** of a function $f : S \to T$ is a right inverse of f: a function $g : T \to S$ such that $fg = \mathrm{id}_T$. Notice that a section need not be an inverse of f because we very well may not have $gf = \mathrm{id}_S$.

Axiom (The axiom of choice)**.** Every surjection has a section.

Exercise 6. Show that the axiom of choice fails in all of the following categories:

(a) topological spaces (i.e., there exists a continuous surjection without a continuous section),

(b) groups (there exists a surjective group homomorphism without a homomorphic section),

(c) abelian groups,

(d) rings,

(e) commutative rings,

(f) the category whose objects are pairs (S, φ) where S is a set and φ is a bijection from S to itself. (Hint: Let S be the set $\mathbf{Z}$ and $\varphi(n) = n + 1$. Let S' be a point (and φ' the only bijection from S' to itself). There is a unique map $(S, \varphi) \to (S', \varphi')$ but this does not have a section.)

Here are several other formulations of the axiom of choice:

Axiom (The axiom of choice)**.** If S_i, $i \in I$ is a family of non-empty sets then $\prod_{i\in I} S_i \neq \varnothing$.

Exercise 7. Prove that the two formulations of the axiom of choice given above are equivalent.

Lemma 4.1 (Zorn's lemma)**.** If S is a partially ordered set in which each totally ordered subset has an upper bound then S contains maximal elements.

Proof. Suppose not. Then if T is a totally ordered subset of S there is an upper bound for T that is not in T. Let f be a function that assigns to each totally ordered subset T of S an upper bound for T not in T. Such a function exists by the axiom of choice.

Define a transfinite sequence of totally ordered subsets of S as follows:

(i) let $T_0 = \varnothing$,

(ii) if $n = m + 1$, let $T_n = T_m \cup \{f(T_m)\}$,

(iii) if n is a limit ordinal, let $T_n = \bigcup_{m<n} T_m$.

The number of T_n will eventually exceed the cardinality of the set of all subsets of S, but there just aren't that many totally ordered subsets of S. This is a contradiction, so there must be some totally ordered subset T of S for which every upper bound for T is contained in T. This upper bound is therefore a maximal element of S (if it weren't maximal, there would be an upper bound for T not contained in T). $\square$

4.2 Ultrafilters

Let X be a set. A **filter** in X is a generalization of a subset of X.

Definition 4.2. A *filter* is a collection F of subsets of X satisfying the following properties:

FIL1 $\varnothing \notin F$,

FIL2 if $U \subset V \subset X$ and $U \in F$ then $V \in F$, and

FIL3 if $U, V \in F$ then there is a $W \in F$ with $W \subset U \cap V$.

If in addition F satisfies

FIL4 F is maximal with respect to properties **FIL1**, **FIL2**, and **FIL3**

then F is called an **ultrafilter**.

We have used a slightly non-standard axiomatization of the notion of an ultrafilter so that generalizations will be straightforward.

Exercise 8. (a) Let F be a filter of X. Show that if $U, V \in F$ then $U \cap V \in F$.

(b) Let F be an ultrafilter of X and $A \subset X$ a subset. Show that $A \in F$ or $(X \smallsetminus A) \in F$.

Proposition 4.3. Every filter of X is contained in an ultrafilter.

Proof. Let F be a filter and let C be the collection of all filters of X containing F, partially ordered by inclusion. An ascending union of filters is a filter, so C contains maximal elements. Any such maximal element is an ultrafilter containing F. □

Proposition 4.4. Let F be a collection of subsets of a set X. Then F is contained in a filter and only if F satisfies the following property: If $U_1, \ldots, U_n \in F$ is a finite collection of elements of F then $\bigcap_i U_i \neq \varnothing$.

Proof. Let F' be the collection of all finite intersections of elements of F and let F'' be the collection of all $V \subset X$ that contain some element of F'. We verify that F'' is a filter.

(i) If $\varnothing \in F''$ then $\varnothing \supset \bigcap_i U_i$ for some finite collection of $U_i \in F$, but this is contrary to our assumption on F.

(ii) If $U \subset V \subset X$ and $U \in F''$ then U contains W for some $W \in F'$. Therefore V also contains W so $V \in F''$.

(iii) If $U, V \in F''$ then U contains U' for some $U' \in F'$ and V contains V' for some $V' \in F'$. Therefore $U \cap V$ contains $U' \cap V'$, which is in F' because F' is closed under finite intersections (by definition).

□

Corollary 4.5. A filter is contained in an ultrafilter if and only if it satisfies the hypothesis of Proposition 4.4.

Exercise 9. Let X be a set.

(a) Let S be a subset of X. Define $F_S = \{S' \subset X \big| S' \supset S\}$. Show that F is a filter.

(b) Let x be a point of X. Show that $F_{\{x\}}$ is an ultrafilter.

Exercise 10. Show that an arbitrary intersection of filters is a filter.

4.2.1 Functoriality

Let $f : X \to Y$ be a function. Let F be an ultrafilter of X. Define

$$f(F) = \big\{S \subset Y \big| f^{-1}(S) \in F\big\}.$$

Exercise 11. Show that if F is a filter, so is $f(F)$, and that if F is an ultrafilter, so is $f(F)$.

Now suppose that $f : X \to Y$ is a function and F is a filter of Y. Define

$$f^{-1}(F) = \big\{S \subset X \big| S \supset f^{-1}(T) \text{ for some } T \in F\big\}.$$

Exercise 12. Show that if F is a filter then $f^{-1}(F)$ is a filter.

4.2.2 Convergence, separation and compactness

Let X be a topological space. An ultrafilter F of X is said to converge to $x \in X$ if, for every open $U \subset X$ with $x \in U$, we have $U \in F$.

Proposition 4.6. A topological space X is Hausdorff if and only if each ultrafilter has at most one limit.

Proof. Suppose that X is Hausdorff and F is an ultrafilter in X with limits x and y. Then for all open neighborhoods $x \in U$ and $y \in V$ of x and y, we have $U, V \in F$. Therefore $U \cap V \in F$ so $U \cap V \neq \varnothing$. That is, it is impossible to find open neighborhoods U and V of x and y such that $U \cap V = \varnothing$. Since X is Hausdorff, this means $x = y$.

Suppose now that each ultrafilter has at most one limit. Let x and y be points of X such that every neighborhood of x meets every neighborhood of y; let E be the collection of all subsets of X that are an open neighborhood of *either* x or y. Then by Corollary 4.5, E is contained in an ultrafilter. But any ultrafilter containing E will converge to both x and to y. Therefore $x = y$. Thus if x and y are distinct there must be open neighborhoods of x and y that do not meet; that is, X is Hausdorff. □

Proposition 4.7. A topological space X is compact if and only if each ultrafilter in X has *at least* one limit.

Proof. Let X be a compact topological space and F an ultrafilter. If F has no limit then for every x there is an open subset $U_x \subset X$ with $U_x \notin F$. Since F is an ultrafilter, this means that $Z_x = X \setminus U_x$ must be in F. Since X is compact, there is a finite subset $x_1, \dots, x_n \in X$ such that the U_{x_i} cover X. Therefore, $\bigcap_{i=1}^n Z_{x_i}$ is empty. But $\bigcap_{i=1}^n Z_{x_i}$ is in F (it is a finite intersection of elements of F). Therefore $\varnothing \in F$, so F could not have been a filter.

Conversely, suppose that every ultrafilter has a limit and $X = \bigcup_{i \in I} U_i$ is a cover with no finite subcover. Let $Z_i = X \setminus U_i$ and let E be the collection of all Z_i. Note that finite intersections of the Z_i are non-empty because if $\bigcap_{j=1}^n Z_{i_j} = \varnothing$ we would have $\bigcup_{j=1}^n U_{i_j} = X$. Thus, E is contained in some ultrafilter F (by Corollary 4.5). But by assumption F has a limit x. This means that every open neighborhood of x lies inside F, including some whichever U_i contain x (there must be at least one since $x \in \bigcup_i U_i$). Therefore both U_i and Z_i are in F for some i, which means that $\varnothing = U_i \cap Z_i \in F$, contradictory to F being a filter. □

4.3 Tychonoff's theorem

Let I be a set and suppose that for each $i \in I$ we have a topological space X_i. Let $X = \prod X_i$ and let $p_i : X \to X_i$ be the projections.

Exercise 13. Verify that if F is an ultrafilter of X then F converges to $x \in X$ if and only if $p_i(F)$ converges to $p_i(x)$ for all i.

Theorem 4.8 (Tychonoff)**.** If X_i is compact for every i then $\prod_{i\in I} X_i$ is compact.

Proof (Cartan, Bourbaki). Let $X = \prod X_i$ and let $p_i : X \to X_i$ be the projections. We would like to show that every ultrafilter of X converges to some $x \in X$. But if F is an ultrafilter, the set of limits of F is $\prod_{i\in I} S_i$ where S_i is the set of limits of $p_i(F)$: indeed, by the exercise, $x \in X$ is a limit of F if and only if $p_i(x)$ is a limit of $p_i(F)$ for all i. Since each X_i is compact, it follows that $S_i \neq \varnothing$ for all i. Therefore the axiom of choice implies $\prod_{i\in I} S_i \neq \varnothing$. $\square$

Week 5: Function spaces

5.1 Topologies on the set of continuous functions between two topological spaces

Definition 5.1. Let X and Y be topological spaces and let $\mathrm{Cont}(Y, X)$ be the set of continuous functions from Y to X. The **compact-open** topology on $\mathrm{Cont}(Y, X)$ is the coarsest topology in which the sets

$$S(C, U) = \left\{f : Y \to X \middle| f \text{ continuous and } f(C) \subset U\right\}$$

are open for every $C \subset Y$ compact and $U \subset X$ open.

Definition 5.2. Let X be a metric space and Y a topological space. The **compact convergence** topology is the coarsest topology in which the sets

$$B_C(f, \epsilon) = \left\{g : Y \to X \middle| g \in \mathrm{Cont}(Y, X) \text{ and } d_C(f, g) < \epsilon\right\}$$
$$d_C(f, g) = \sup\left\{d(f(y), g(y)) \middle| y \in C\right\}$$

are open.

The following notation is convenient: if X is a metric space and $Z \subset X$ is a subset, let

$$B(C, \epsilon) = \left\{x \in X \middle| d(x, f(y)) < \epsilon \text{ for some } y \in C\right\}.$$

Lemma 5.3. Let Z be a compact subset of a metric space X and suppose that $Z \subset U$ for some open $U \subset X$. Then $B(Z, \epsilon) \subset U$ for some $\epsilon > 0$.

Proof. For each $x \in Z$ there is an $\epsilon(x)$ such that $B(x, \epsilon(x)) \subset U$ by definition of the metric topology. Then the balls $B(x, \frac{\epsilon(x)}{2})$ cover Z so finitely many of them suffice—say $x_1, \ldots, x_n$. If $y \in Z$ then $d(y, x_i) < \frac{\epsilon}{2}$ for some i, so $y \in B(x_i, \frac{\epsilon(x_i)}{2})$ for some i.

Let $\epsilon = \frac{1}{2}\min\{\epsilon(x_i)\}$. Then if $y \in Z$ and $z \in B(Z, \epsilon)$ then $d(x, y) < \epsilon$. We can find x_i such that $d(y, x_i) < \frac{\epsilon(x_i)}{2}$ because the $B(x_i, \epsilon(x_i)/2)$ cover Z. Therefore

$$d(z, x_i) < \frac{\epsilon}{2} + \frac{\epsilon(x_i)}{2} \leq \epsilon(x_i)$$

so $z \in B(x_i, \epsilon(x_i)) \subset U$. Therefore $B(Z, \epsilon) \subset \bigcup_{i=1}^n B(x_i, \epsilon(x_i)) \subset U$. $\square$

Proposition 5.4. If Y is a topological space and X is a metric space, the compact convergence topology and the compact-open topology on $\mathrm{Cont}(Y, X)$ coincide.

Proof. First we show that if $C \subset Y$ compact and $U \subset X$ open then $S(C, U)$ is open in the compact convergence topology. We show that for any $f \in S(C, U)$ there is an $\epsilon > 0$ such that $B_C(f, \epsilon) \subset S(C, U)$. By the lemma, we can find $\epsilon > 0$ such that $B(f(C), \epsilon) \subset U$. But then if $g \in B_C(f, \epsilon)$ then $d(g(y), f(y)) < \epsilon$ for all $y \in C$ so $g(y) \in B(f(C), \epsilon) \subset U$ for all $y \in C$. That is $g \in S(C, U)$.

Now we show that $B_C(f, \epsilon)$ is open in the compact open topology for all continuous $f : Y \to X$, all $\epsilon > 0$, and all compact $C \subset Y$. It will be enough to demonstrate that there is a $W \subset \mathrm{Cont}(Y, X)$ that is open in the compact open topology and satisfies $f \in W \subset B_C(f, \epsilon)$.

Cover $f(C)$ by balls $V_i = B(x_i, \frac{\epsilon}{3})$, each of diameter $\frac{\epsilon}{3}$. Let $U_i = B(x_i, \frac{\epsilon}{2})$. Since $f(C)$ is compact, only finitely many of the V_i—say $V_1, \ldots, V_n$—suffice to cover $f(C)$. Let

$$W = \bigcup_{i=1}^{n} S(f^{-1}(\overline{V}_i) \cap C, U_i) \subset \mathrm{Cont}(Y, X).$$

Note that $f \in W$ because $f(f^{-1}(\overline{V}_i) \cap C) \subset V_i \subset U_i$ by definition. Note also that $f^{-1}(\overline{V}_i) \cap C$ is closed in C, hence compact, so $S(f^{-1}(\overline{V}_i) \cap C, U_i)$ is open in the compact open topology for each i. It follows therefore that W is the intersection of open subsets in the compact open topology, hence is open in the compact open topology.

If $g \in W$ then $g(V_i) \subset U_i$ for all i. Therefore, if $y \in V_i$, both $f(y)$ and $g(y)$ will be in U_i. It follows that

$$d(f(y), g(y)) \leq d(f(y), x_i) + d(x_i, g(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since every $y \in C$ is contained in $f^{-1}(V_i)$ for some i, this means that $d(f(y), g(y)) < \epsilon$ for all $y \in C$. Therefore $W \subset B_C(f, \epsilon)$. □

5.2 The universal property of the mapping space

Theorem 5.5. Let X, Y, and Z be topological spaces. If Y has a basis of compact subsets then

$$\mathrm{Cont}(Z, \mathrm{Cont}(Y, X)) = \mathrm{Cont}(Z \times Y, X)$$

when $\mathrm{Cont}(Y, X)$ is given the compact-open topology.

First, note that there is a natural identification

$$\mathrm{Func}(Z, \mathrm{Func}(Y, X)) = \mathrm{Func}(Z \times Y, X).$$

If $g : Z \times Y \to X$ is a function, we get a function $f : Z \to \mathrm{Func}(Y, X)$ by defining $f(z)(y) = g(z, y)$. Likewise, if $f : Z \to \mathrm{Func}(Y, X)$ is given, we can take $g(z,y) = f(z)(y)$. These are obviously inverse constructions. We will verify that g is continuous if and only if the corresponding function f takes values in $\mathrm{Cont}(Y,X)$ and is continuous.

The proof has two parts, which we state as two lemmas:

Lemma 5.6. Let X, Y, and Z be topological spaces. If $g : Z \times Y \to X$ is continuous then f takes values in $\mathrm{Cont}(X,Y)$ and f is continuous.

Proof. If g is continuous then for each $z \in Z$ the function $f(z) : Y \to X$ is continuous (because the composition $Y \cong \{z\} \times Y \to Z \times Y \to X$ is continuous). Therefore $f(z) \in \mathrm{Cont}(Y,X)$. We check that f is continuous.

Suppose $f(z) \in S(C,U)$ for some $C \subset Y$ compact and $U \subset X$ open. Then since g is continuous, $g^{-1}(U) \subset Z \times Y$ is continuous. Therefore for each $y \in C$ there is an open $V_y \subset Z$ containing z and $W_y \subset Y$ containing y such that $V_y \times W_y \subset g^{-1}(U)$. The W_y cover C and C is compact so C is already contained in $\bigcup_{i=1}^n W_{y_i}$. Take $V = \bigcap_{i=1}^n V_{y_i}$. Then $z \in V$ and V is open. Furthermore, $V \subset f^{-1}S(C,U)$, for if $z' \in V$ then $g(z', W_{y_i}) \subset U$ for all y_i so in particular $g(z', C) \subset U$, i.e., $z' \in S(C,U)$. Therefore $f^{-1}S(C,U)$ is open in $Z \times Y$. This proves f is continuous. □

Lemma 5.7. Suppose Y has a neighborhood basis of compact subsets (that is, every open neighborhood of any $y \in Y$ contains a compact neighborhood of y).[1] If f is continuous with values in $\mathrm{Cont}(Y,X)$ then g is continuous.

Proof. Suppose that $U \subset X$ is open and $g(z,y) \in U$. Since $f(z)$ is continuous, the subset $f(z)^{-1}(U) \subset Y$ is open (and contains y). Because Y has a neighborhood basis of compact subsets, we can find a compact neighborhood V of y contained in $f(z)^{-1}(U)$. Then $S(V,U)$ is an open subset of $\mathrm{Cont}(Y,X)$ so $f^{-1}(S(V,U)) \subset Z$ is open (and notice that it contains z because $g(z,V) \subset U$). Let $W \subset V$ be an open neighborhood of y. Then for any $(z', y') \in f^{-1}S(V,U) \times W$ we have

$$g(z', y') = f(z')(y') \in S(V,U)(W) \subset S(V,U)(V) \subset U.$$

That is $f^{-1}S(V,U) \times W$ is an open neighborhood of (z,y) that is contained in $g^{-1}(U)$. Since we can find such a neighborhood for any (z,y), it follows that $g^{-1}(U)$ is open in $Z \times Y$, which means that g is continuous. □

Together, the two lemmas prove the theorem.

[1] Perhaps it is appropriate here to recall the definition of a neighborhood. A subset V of Y containing y is said to be a neighborhood of y if there is an open $U \subset Y$ such that $y \in U \subset V$. Note that this is not the same as the definition in [Mun], where neighborhoods are assumed to be open.

Week 6: Urysohn's Lemma

Week 7: Homotopy and the fundamental group

Week 8: Covering spaces

Exercise 14. Let B be a topological space and let X be the subset of $\mathrm{Cont}(I,B) \times \mathrm{Cont}(I,B)$ consisting of those pairs (f,g) such that $f(1) = g(0)$. The map $\mu : X \to \mathrm{Cont}(I,B)$ defined by

$$\mu(f,g)(t) = \begin{cases} f(2t) & t \le \frac{1}{2} \\ g(2t-1) & t \ge \frac{1}{2} \end{cases}$$

is continuous.

Exercise 15. Let $p : (E,e) \to (B,b)$ be a covering space. Show that the induced map $\mathrm{Cont}((I,0),(E,e)) \to \Omega(B,b)$ is a *homeomorphism.*

Proposition 8.1. Suppose that B is locally contractible. Then for any basepoint $b \in B$ the map $\Omega(B,b) \to \pi_1(B,b)$ is continuous.

Proof. We have to show that for $\gamma \in \pi_1(B,b)$, the set of $\beta \in \Omega(B,b)$ that are homotopic to γ is open. Cover B by contractible open sets U. Since $\gamma : I \to B$ is continuous and I is compact, we can find a sequence of $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each i, we have $\gamma([t_{i-1}, t_i]) \subset U_i$ for some contractible $U_i \subset B$. For each i, let V_i be a contractible neighborhood of $\gamma(t_i)$ in $U_i \cap U_{i+1}$. Consider the open set

$$\begin{aligned} W = S([t_0,t_1],U_1) \cap S([t_1,t_2],U_2) \cap \cdots \cap S([t_{n-1},t_n],U_n) \\ \cap S(\{t_1\},V_1) \cap \cdots \cap S(\{t_n\},V_n) \cap \Omega(B,b) \subset \Omega(B,b). \end{aligned}$$

Suppose $\beta \in W$. We will show that $\beta \simeq \gamma$. We construct the homotopy $H : I \times I \to B$ as follows: first, define

$$\begin{aligned} H(0,t) &= \beta(t) && \text{for all } t \in [0,1], \\ H(1,t) &= \gamma(t) && \text{for all } t \in [0,1], \\ H(s,0) &= H(s,1) = b && \text{for all } s \in [0,1]. \end{aligned}$$

Second, for each t_i, choose a path h_i from $\beta(t_i)$ to $\gamma(t_i)$ inside of V_i (which exists because V_i is contractible). Let $H(s,t_i) = h_i(s)$. Now, for each i we have defined H on the boundary of $[0,1] \times [t_{i-1}, t_i]$ to take values inside of U_i. Since U_i is contractible, it is possible to extend this map to a continuous map $H_i : [0,1] \times [t_{i-1}, t_i] \to U_i$. These maps all agree where they overlap, so we get a continuous map $H : I \times I \to B$ yielding a homotopy between β and γ. $\square$

Exercise 16. Show that in the proposition above, it would have been enough to assume that B be **locally path connected** and **semilocally simply connected**.

Recall that $\Omega(B, b) = \mathrm{Cont}((S^1, *), (B, b))$ is the based loop space of (B, b). This is not quite a group, but is very close: if $f, g \in \Omega(B, b)$ we can define

$$f \cdot g(t) = \begin{cases} f(2t) & t \leq \frac{1}{2} \\ g(2t-1) & t \geq \frac{1}{2} \end{cases}$$
$$f^{-1}(t) = f(1-t)$$

Let $*$ denote the constant loop $S^1 \to B$ with value b. A **continuous right action** of $\Omega(B, b)$ on a set S is a *continuous* function $S \times \Omega(B, b) \to S : (x, f) \mapsto x.f$ such that

(a) $x.* = x$ for all $x \in S$, and

(b) $x.(fg) = (x.f).g$ for all $f, g \in \Omega(B, b)$.

Observe that giving an action of $\Omega(B, b)$ on S induces an action of $\pi_1(B, b)$ on S. Does the converse hold? Does every $\pi_1(B, b)$-action on S induce an $\Omega(B, b)$-action on S?

Theorem 8.2. Suppose that (B, b) is a path connected, locally contractible space and we are given a continuous right action of $\Omega(B, b)$ on a set S. Then there is a covering space $p : E \to B$ and a $\Omega(B, b)$-equivariant bijection between $p^{-1}(b)$ and S.

Proof. Assume that we are given a right action of $\Omega(B, b)$ on a set S. We will construct a covering space $p : E \to B$ with fiber S.

Let $P(B, b) = \mathrm{Cont}((I, 0), (B, b))$ be the based path space of B. Define an equivalence relation on $S \times P(B, b)$: say that $(x, f) \sim (y, g)$ if $y = x.fg^{-1}$. Let E be the quotient of $P(B, b) \times S$ by this equivalence relation. Give E the quotient topology.

We have a function $P(B, b) \times S \to B$ sending (x, f) to $f(1)$. This is a well-defined function on equivalence classes, so it induces a function $p : E \to B$, which must be continuous by definition of the quotient topology. We check that this is a covering space.

We show that $p^{-1}(U) \cong U \times S$ whenever U is a contractible subset of B. Since B is locally contractible, B has a cover by contractible subsets, so this will show that E is a covering space of B.

We need to construct a continuous function $\varphi : p^{-1}(U) \to S$. We cannot simply use the function $\varphi(x, f) = x$ because this is not well-defined on equivalence classes. Pick a point $c \in U$ and a path f from b to 1. If $(x, g) \in p^{-1}(U)$ then $g(1) \in U$. Choose a path h from $g(1)$ to $f(1)$ (which exists because U is contractible). Then gh and f have the same endpoint so ghf^{-1} is a loop in B based at b. Define $\varphi(x, g) = x.ghf^{-1}$.

We have to check this is well-defined: suppose that $(x, g) \sim (x', g')$; let h' be a path from $g'(1)$ to $f(1)$. Then $g(1) = g'(1)$ (because $(x, g) \sim (x', g')$) and h' is homotopic to h (because U is contractible). We have $x' = x.gg'^{-1}$.

Noting that $gg'^{-1} \simeq ghh'^{-1}g'^{-1} = (gh)(g'h')^{-1}$ we notice that this means $x' = x.gh.(g'h')^{-1}$. Therefore

$$\varphi(x', g') = x'.g'h'f^{-1} = x.gh.(g'h')^{-1}.g'h'.f^{-1} = x.ghf^{-1} = \varphi(x, g).$$

This shows that φ is well-defined. To see that $(\varphi, p) : p^{-1}(U) \to S \times U$ is surjective, pick $(x, d) \in S \times U$. Choose a path h from $c = f(1)$ to d. Then

$$\varphi(x, fh) = x.fh.h^{-1}.f^{-1} = x$$

so $(x, d) = (\varphi(x, fh), p(x, fh))$.

To see that (φ, p) is injective, suppose that $\varphi(x, g) = \varphi(x', g')$. Then $g(1) = g'(1)$ and if h is a path from this point to $f(1)$ then $x.gh.f^{-1} = x'.g'h.f^{-1}$. Therefore $x' = x.gh.f^{-1}.f.(g'h)^{-1} = x.gg'^{-1}$ so $(x, g) \sim (x', g')$. It follows that (φ, p) is a bijection.

Now let's check that φ is continuous. Choose a homotopy H between $\mathrm{id} : U \to U$ and the constant map with value c. This is a continuous function $I \times U \to U$ whose value on $(0, d)$ is d and whose value on $(1, d)$ is c for all $d \in U$. We may also regard this as a continuous map $H : U \to \mathrm{Cont}(I, U)$. Now consider the function $\psi : P(B, b, U) \to \Omega(B, b)$ defined by $\psi(g) = g \cdot H(g(1)) \cdot f^{-1}$. We have

$$\varphi(x, g) = x.\psi(g)$$

and ψ is continuous by Exercise 14. Therefore the map

$$\varphi : S \times P(B, b, U) \to S \times \Omega(B, b) \to S$$

is continuous.

Finally, we also have to check that the map $(\varphi, p) : p^{-1}(U) \to S \times U$ constructed above has a continuous inverse. Let H be as above, and define $\xi : S \times U \to p^{-1}(U)$ as follows:

$$\xi(x, d) = (x, f.H(d)^{-1}).$$

One must check that ξ actually is continuous: ξ is the composition of

$$\begin{aligned} S \times U \to S \times \mathrm{Cont}((I, 0, 1), (B, b, c)) \times \mathrm{Cont}((I, 0, 1), (U, c, U)) \\ \to S \times \mathrm{Cont}((I, 0, 1), (B, b, U)) \to S \times P(B, b) \to E \end{aligned}$$

and all of these maps are continuous (the second one by Exercise 14). We also have to check that (φ, p) and ξ are inverses:

$$\begin{aligned} &\psi(\xi(x, d)) = \psi(x, f.H(d)^{-1}) = (x.f.H(d)^{-1}.H(d).f^{-1}, d) = (x, d) \\ &\xi(\psi(x, g)) = \xi(x.g.H(g(1)).f^{-1}, g(1)) = (x.g.H(g(1)).f^{-1}, f.H(g(1))^{-1}). \end{aligned}$$

I claim that this last term is equivalent to (x, g). Indeed, $g(1)$ is the endpoint of $f.H(g(1))^{-1}$ and we have

$$x.g.(f.H(g(1))^{-1})^{-1} = x.g.H(g(1)).f^{-1}$$

which is the definition of what it means for the point (x, g) to be equivalent to the point $(x.g.H(g(1)).f^{-1}, f.H(g(1))^{-1})$.

This proves that $p : E \to B$ is a covering space. To finish the proof we have to check that we've gotten the right action on $p^{-1}(b)$. Recall that the action of $\Omega(B, b)$ on $p^{-1}(b)$ is defined by $x.\gamma = \widetilde{\gamma}(1)$ where $\widetilde{\gamma} : (I, 0, 1) \to (E, x, p^{-1}(b))$ is a lift of $\gamma : (I, 0, 1) \to (B, b, b)$.

Now, $\widetilde{\gamma}$ is an element of $P(B, b)$ so $(x, \widetilde{\gamma})$ is an element of $S \times P(B, b)$. We have $\widetilde{\gamma}(1) = (x, \widetilde{\gamma}) \sim (x.\widetilde{\gamma}, 1) \in p^{-1}(b)$ by definition of the equivalence relation on $S \times P(B, b)$. Hence the two actions of $\Omega(B, b)$ on $p^{-1}(b)$ are the same. □

The construction above is actually **functorial**.

Theorem 8.3. Assume that B is path connected and locally contractible. There is an equivalence of categories between the category of right $\pi_1(B, b)$-sets and the category of covering spaces of B.

Week 9: Group actions and covering spaces

9.1 The universal cover

Suppose that B is locally contractible and path connected. Let $\pi_1(B, b)$ act on itself. The corresponding cover of B is known as the **universal cover**.

Proposition 9.1. The unviersal cover is simply connected.

Proof. Consider the exact sequence

$$1 \to \pi_1(E, e) \to \pi_1(B, b) \to \pi_0(p^{-1}(b), e).$$

The last map is a bijection because $\pi_0(p^{-1}(b), e) = p^{-1}(b) = \pi_1(B, b)$ by definition. Therefore $\pi_1(E, e) = 1$. □

Direct proof. We can also show this directly. Suppose that γ is in $\pi_1(E, e)$. Then $p\gamma$ is a loop in $\pi_1(B, b)$. Then by definition, if $\beta \in \pi_1(B, b)$ the action of γ on e is $e.p\gamma = e\gamma = \gamma(1)$ (remember that $e \in p^{-1}(b) = \pi_1(B, b)$ by definition). But γ is a loop in $\pi_1(E, e)$ so $\gamma(1) = e$. But then $e\gamma = e$. That is, $\gamma = 1$ since $\pi_1(B, b)$ is a group. □

9.2 Connectedness

Let (B, b) be a pointed space. In this section $p : E \to B$ will be a covering space of B and $S = p^{-1}(b)$. We assume that B is path connected and locally contractible.

Proposition 9.2. E is path connected if and only if the action of $\pi_1(B, b)$ on S is transitive.

Proof. Suppose that E is path connected. Then for every $x, y \in p^{-1}(b)$, there is a path f from x to y. Then pf is a loop in B based at b and $x.pf = y$.

Now suppose that $\pi_1(B, b)$ acts transitively on S. Suppose $x, y \in E$. Then since B is path connected, we can choose paths f from $p(x)$ to b and g from $p(y)$ to b. Lift f and g to paths $\widetilde{f}$ and $\widetilde{g}$ with starting points x and y. Then $\widetilde{f}(1)$ and $\widetilde{g}(1)$ are in S. Therefore there is some loop $h \in \pi_1(B, b)$ with $\widetilde{f}(1).h = \widetilde{g}(1)$. This means that if we lift h to a path $\widetilde{h}$ starting at $\widetilde{f}(1)$ its endpoint is $\widetilde{g}(1)$. Thus the path $\widetilde{f}.\widetilde{h}.\widetilde{g}^{-1}$ is a path from x to y. This shows that E is path connected. $\square$

Proposition 9.3. Let G be a group. There is an equivalence

$$\left\{ (S, x) \,\middle|\, \begin{array}{l} S \text{ is a transitive} \\ \text{right } G\text{-set} \\ \text{and } x \in S \end{array} \right\} \simeq \left\{ \text{subgroups of } G \right\}$$

Proof. The equivalence sends (S, x) to $\mathrm{Stab}(x) \subset G$ and sends $H \subset G$ to $(H\backslash G, H)$. $\square$

Corollary 9.4. There is an equivalence

$$\left\{ \text{connected covers } p : (E, x) \to (B, b) \right\} \simeq \left\{ \text{ subgroups of } \pi_1(B, b) \right\}.$$

What about unpointed covers?

Proposition 9.5. There is an equivalence

$$\left\{ \text{non-empty, transitive actions of } G \right\} \simeq \left\{ \begin{array}{c} \text{conjugacy classes of} \\ \text{subgroups of } G \end{array} \right\}.$$

Proof. Let G act transitively on S. Pick $x \in S$. Let $\Phi(S)$ be the conjugacy class of $\mathrm{Stab}(x)$. Notice that if we were to replace x by y then $y = x.g$ for some $g \in G$ and $\mathrm{Stab}(y) = g^{-1} \mathrm{Stab}(x) g$ so this is well-defined.

Conversely, suppose that H is a subgroup of G. Then $H\backslash G$ is a set with a transitive right G-action. Suppose that H is replaced by $g^{-1}Hg$, which we abbreviate to H^g. Then $H\backslash G$ will be replaced by $H^g\backslash G$. However, $H\backslash G$ and $H^g\backslash G$ are isomorphic as right G-sets: consider the function $\mu : H\backslash G \to H^g\backslash G$ sending Ht to $H^g g^{-1} t = g^{-1}Ht$. We have to check this is well defined, i.e., that if $Ht = Ht'$ then $\mu(Ht) = \mu(Ht')$. If $Ht = Ht'$ then $t' = ht$ for some $h \in H$ so

$$\mu(Ht') = H^g g^{-1} ht = g^{-1}Hht = g^{-1}Ht = \mu(Ht).$$

It is evident that the function $\nu : H^g\backslash G \to H\backslash G$ defined by $\nu(H^g t) = Hgt$ is inverse to μ. Furthermore, μ is equivariant, since we have $\mu(Htu) = H^g g^{-1} tu = \mu(Ht).u$. Thus if we define $\Psi(H) = H\backslash G$, this gives a well-defined map from the set of conjugacy classes of subgroups of G to isomorphism classes of non-empty, transitive actions of G.

We can check these constructions are inverse to each other by noting that $\mathrm{Stab}(x)\backslash G$ is isomorphic to S and the stabilizer of the point of $H\backslash G$ corresponding to H is precisely H. $\square$

Corollary 9.6. There is an equivalence

$$\Big\{\text{connected covers } p : E \to B\Big\} \simeq \Big\{\text{conjugacy classes of subgroups of } \pi_1(B, b)\Big\}.$$

9.3 Automorphisms

The **automorphism group** of a covering space $p : E \to B$ is the set of continuous maps $f : E \to E$ such that $pf = p$. We denote this group by $\mathrm{Aut}_B(E)$.

Suppose that $p : E \to B$ corresponds to the right action of $\pi_1(B, b)$ on a set S. Then since covering spaces of B form an equivalent category to the category of right actions of $\pi_1(B, b)$, this means that $\mathrm{Aut}_B(E) = \mathrm{Aut}_G(S)$ where $\mathrm{Aut}_G(S)$ is the set of **equivariant** bijections from S to itself. Recall that a function $f : S \to S$ is called equivariant if $f(x.g) = f(x).g$ for every $g \in G$.

Proposition 9.7. Suppose that G acts transitively on a set S and x is a point of S. Then $\mathrm{Aut}_G(S) \cong N(\mathrm{Stab}(x))/\mathrm{Stab}(x)$ where $N(\mathrm{Stab}(x))$ is the normalizer of the stabilizer of x.

Proof. First we define a function $\mu : \mathrm{Aut}_G(S) \to N(\mathrm{Stab}(x))/\mathrm{Stab}(x)$. Suppose $\varphi : S \to S$ is an equivariant bijection. Since G acts transitively on S, we have $\varphi(x) = x.g$ for some g in G. On the other hand, φ is supposed to be equivariant, so $\varphi(x.h) = \varphi(x).h = x.gh$ for all $h \in G$. But if $h \in \mathrm{Stab}(x)$ then we have $\varphi(x.h) = \varphi(x) = x.g$ and $\varphi(x.h) = \varphi(x).h = x.gh$. Therefore $x.gh = x.g$, or effectively, $x.ghg^{-1} = x$. Thus $ghg^{-1} \in \mathrm{Stab}(x)$, so $g \in N(\mathrm{Stab}(x))$. Define $\mu(\varphi)$ to be the class of g in $N(\mathrm{Stab}(x))/\mathrm{Stab}(x)$.

Note that g was not unique above, so we have to check this is well defined. Suppose that $\varphi(x) = x.g'$ as well. Then $x.g'g^{-1} = x$ so $g'g^{-1} \in \mathrm{Stab}(x)$. That is g' and g define the same class in $N(\mathrm{Stab}(x))/\mathrm{Stab}(x)$.

Now we construct the inverse $\nu : N(\mathrm{Stab}(x))/\mathrm{Stab}(x) \to \mathrm{Aut}_G(S)$. Suppose $g \in N(\mathrm{Stab}(x))$. Since the action of G on S is transitive, every element of S can be written as $x.h$ for some (possibly non-unique) $h \in G$. Define $\nu(g) : S \to S$ to be the function $\nu(g)(x.h) = x.gh$. We have to check this is a well-defined function: if $x.h = x.h'$ we have $\nu(g)(x.h) = x.gh$ and $\nu(g)(x.h') = x.gh'$. But $h'h^{-1} \in \mathrm{Stab}(x)$ so $gh'h^{-1}g^{-1} \in \mathrm{Stab}(x)$ (because $g \in N(\mathrm{Stab}(x))$), which implies that $x.gh'h^{-1}g^{-1} = x$. That is, $x.gh' = x.gh$. So $\nu(g)(x.h) = \nu(g)(x.h')$. Therefore this is well-defined on $N(\mathrm{Stab}(x))$. We also have to check that $\nu(\mathrm{Stab}(x))$ is the identity automorphism of S. But if $g \in \mathrm{Stab}(x)$ then $\nu(g)(x.h) = x.gh = x.h$ since $x.g = x$; therefore $\nu(g) = \mathrm{id}_S$.

I omit the verification that ν and μ are inverse functions, as well as the explicit verification that they are homomorphisms, both of which are trivial. □

9.4 Symmetry

A cover $p : E \to B$ is called **symmetric** or **regular** or **Galois** if for each $x, y \in \pi^{-1}(b)$ there is an automorphism $\varphi : E \to E$ of the cover such that $\varphi(x) = y$.

Lemma 9.8. A cover $p : E \to B$ is symmetric if and only if the action of $\pi_1(B, b)$ on $p^{-1}(b)$ is symmetric, in the sense that for all $x, y \in p^{-1}(b)$ there is an automorphism ψ of $p^{-1}(b)$ and a $\pi_1(B, b)$-set such that $\psi(x) = y$.

We will call the $\pi_1(B, b)$-sets satisfying the above **symmetric** $\pi_1(B, b)$-sets.

Proposition 9.9. There is an equivalence

$$\left\{ (S, x) \middle| \begin{array}{l} S \text{ a symmetric,} \\ \text{transitive } G\text{-set} \\ x \in S \end{array} \right\} \simeq \Big\{ \text{normal subgroups of } G \Big\}.$$

Proof. Suppose that S is symmetric and $x \in S$. Then the corresponding subgroup of G is $\mathrm{Stab}(x)$. Suppose that $g \in G$. Then there is a G-equivariant bijection $\varphi : S \to S$ such that $\varphi(x) = x.g$. Then

$$\begin{aligned} \mathrm{Stab}(\varphi(x)) &= \big\{ h \in G \big| \varphi(x).h = \varphi(x) \big\} \\ &= \big\{ h \in G \big| \varphi(x.h) = \varphi(x) \big\} \\ &= \big\{ h \in G \big| x.h = x \big\} \\ &= \mathrm{Stab}(x) \end{aligned}$$

since φ is a bijection. On the other hand, $\mathrm{Stab}(x.g) = g^{-1} \mathrm{Stab}(x) g$. This proves $\mathrm{Stab}(x)$ is normal.

Conversely, suppose that $H \subset G$ is normal. Then the corresponding action of G is the right action on $H\backslash G$. Let Hx and Hy be any two elements of $H\backslash G$. Let $g = yx^{-1}$ so that $gx = y$. Then define a map $\varphi : H\backslash G \to H\backslash G$ by $\varphi(Hz) = g^{-1}Hz$. This is G-equivariant because $\varphi(Hz.g') = g^{-1}Hzg' = \varphi(Hz).g'$. Furthermore, $\varphi(Hx) = g^{-1}Hx = Hgx = Hy$. Thus the action of G on $H\backslash G$ is symmetric. $\square$

Corollary 9.10. There is a bijection of sets

$$\Big\{ \text{symmetric, transitive right } G\text{-actions} \Big\} \simeq \Big\{ \text{normal subgroups of } G \Big\}$$

Corollary 9.11. There is an equivalence of categories

$$\left\{ \begin{array}{c} \text{regular covers} \\ p : (E, e) \to (B, b) \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{normal subgroups} \\ \text{of } G \text{ up to} \\ \text{conjugation} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{quotient groups} \\ \text{of } G \text{ up to} \\ \text{conjugation} \end{array} \right\}$$

There is a bijection of sets

$$\left\{ \begin{array}{c} \text{regular covers} \\ p : E \to B \\ \text{up to isom.} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{normal subgroups} \\ \text{of } G \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{quotient groups} \\ \text{of } G \end{array} \right\}$$

Corollary 9.12. Let $p : E \to B$ be a connected covering space corresponding to a normal subgroup $N \subset \pi_1(B, b)$. Then $\mathrm{Aut}_B(E) \cong \pi_1(B, b)/N$.

Proof. Apply Proposition 9.7. $\square$

Week 10: The van Kampen theorem

10.1 Gluing covers

Suppose that $B = U \cup V$ with U and V being open subsets of B, and let $W = U \cap V$. Assume that the basepoint b is in W. To specify a covering space of B we can give a covering space E_U of U, a covering space E_V of V, and an isomorphism of the covering spaces $p_U^{-1}(W)$ and $p_V^{-1}(W)$ of W. Given these data, we can construct a covering space E of B by gluing together these covering spaces.

Let us make this construction precise. Suppose that E is a covering space of B. Let $\Phi(E) = (E_U, E_V, \alpha)$ where α is the isomorphism between the covering spaces $\alpha : p_U^{-1}(E_U) \cong E_W \cong p_V^{-1}(E_V)$.

Suppose that (E_U, E_V, α) is a triple consisting of a cover $p_U : E_U \to U$, a cover $p_V : E_V \to V$, and an isomorphism $\alpha : p_U^{-1}(W) \cong p_V^{-1}(W)$ of covers of W. Then let $\Psi(E_U, E_V, \alpha)$ be the space obtained by dividing $E_U \amalg E_V$ by the equivalence relation $x \sim \alpha(x)$ for $x \in p_U^{-1}(W)$. We get a projection $p : E \to B$ by defining $p(x) = p_U(x)$ if $x \in E_U$ and $p(x) = p_V(x)$ if $x \in E_V$. This is well defined because if $x \sim y$ then $y = \alpha(x)$ so $p(x) = p_U(x) = p_V(\alpha(x)) = p_V(y) = p(y)$. Furthermore, the universal property of the quotient topology guarantees that this projection is continuous.

Lemma 10.1. The projection $p : E \to B$ defined above is a covering space.

Proof. Notice that $p^{-1}(U) = E_U$ and $p^{-1}(V) = E_V$. Since $p_U : E_U \to U$ and $p_V : E_V \to V$ are covering spaces, we can cover U by open sets U_i and V by open sets V_i such that $p_U^{-1}(U_i) = S \times U_i$ and $p_V^{-1}(V_i) = S' \times V_i$ for some sets S and S'. Furthermore, $b \in U$ and $b \in V$ so $p^{-1}(b) = p_U^{-1}(b) \cong S$ and $p^{-1}(b) = p_V^{-1}(B) \cong S'$, so we have a bijection between S and S'. Therefore the U_i and V_i together form a cover of B such that $p^{-1}(U_i) \cong S \times U_i$ and $p^{-1}(V_i) \cong S \times V_i$. $\square$

Exercise 17. Suppose that B is a connected topological space and $p : E \to B$ is a continuous map. Assume that B has a cover by open sets U_i such that $p\big|_{U_i} : p^{-1}(U_i) \to U_i$ is a covering space for each i. Show that under this hypothesis, $p : E \to B$ is a covering space.

We are going to show that to give a covering space of B is essentially equivalent to giving covering spaces of U and V that agree over $U \cap V$. But what does it mean to say these are equivalent? It means first of all that every triple (E_U, E_V, α) consisting of a covering space $p_U : E_U \to U$, a covering space $p_V : E_V \to V$, and an isomorphism $\alpha : p_U^{-1}(W) \to p_V^{-1}(W)$ of covering spaces of W, comes from some covering space $p : E' \to B$, in the sense that there are isomorphisms $f_U : E_U \cong E'_U$ and $f_V : E_V \cong E'_V$ such that if $x \in p_U^{-1}(W)$ then $f_U(x) = f_V(\alpha(x))$.

It also means that to give a morphism of covering spaces $f : E \to E'$ of B it is the same as to give morphisms of covering spaces $f_U : E_U \to E'_U$ and

$f_V : E_V \to E'_V$ such that f_U and f_V agree where their definitions overlap: $f_U\big|_{E_W} = f_V\big|_{E_W} : E_W \to E'_W$.

Proposition 10.2. The constructions Φ and Ψ determine an equivalence of categories:

$$\Big\{\text{covers of } B\Big\} \simeq \left\{ (E_U, E_V, \alpha) \,\middle|\, \begin{array}{l} E_U \text{ covers } U \\ E_V \text{ covers } V \\ \alpha : p_U^{-1}(W) \cong p_V^{-1}(W) \end{array} \right\}.$$

That is $\Phi(\Psi(E_U, E_V, \alpha)) \cong (E_U, E_V, \alpha)$ and $\Psi(\Phi(E)) \cong E$.

Proof. First we show that maps of covering spaces $E \to E'$ of B correspond to maps $f_U : E_U \to E'_U$ and $f_V : E_V \to E'_V$ that agree on E_W. Indeed, if we are f_U and f_V then we get maps $f'_U : E_U \to E'$ and $f'_V : E_U \to E'$ that agree on E_W. But E is the union of the open sets E_U and E_V so f'_U and f'_V glue together to give a continuous map of covering sapces $E \to E'$.

Now we show that every covering space $p : E \to B$ comes from some (E'_U, E'_V, α). Indeed, we can take $E'_U = p^{-1}(U)$ and $E'_V = p^{-1}(V)$. Then $p_U^{-1}(W) = p_V^{-1}(W)$ so we can take α to be the identity map. □

10.2 Gluing fundamental groups

Suppose, as in the last section, that $B = U \cup V$, define $W = U \cap V$, and assume that $b \in W$. We have a map

$$\pi_1(U, b) \underset{\pi_1(W,b)}{*} \pi_1(V, b) \to \pi_1(B, b)$$

because we have maps $\pi_1(U, b) \to \pi_1(B, b)$ and $\pi_1(V, b) \to \pi_1(B, b)$ that agree on $\pi_1(W, b)$. In this section, we will prove that this homomorphism is an isomorphism.

Theorem 10.3 (Siefert, van Kampen)**.** Suppose that $B = U \cup V$ for open subsets U and V of B both containing the basepoint b. Let $W = U \cap V$. Assume that U, V, and W are all path connected and locally contractible. Then the map

$$\pi_1(U, b) \underset{\pi_1(W,b)}{*} \pi_1(V, b) \to \pi_1(B, b)$$

is an isomorphism of groups.

Now, assume that U, V, and W are all path connected and locally contractible. Then we can identify

$$\begin{aligned} \mathrm{Cov}(U) &\simeq \pi_1(U, b)\text{-}\mathsf{Sets} \\ \mathrm{Cov}(V) &\simeq \pi_1(V, b)\text{-}\mathsf{Sets} \\ \mathrm{Cov}(W) &\simeq \pi_1(W, b)\text{-}\mathsf{Sets}. \end{aligned}$$

Define $G = \pi_1(U, b) *_{\pi_1(W,b)} \pi_1(V, b)$. A G-set consists of a set S with actions of $\pi_1(U, b)$ and of $\pi_1(V, b)$ that agree on $\pi_1(W, b)$. Since a $\pi_1(U, b)$-set is the same as a covering space of U, etc., this means that a G-set consists of a covering space of $p_U : E_U \to U$, a covering space $p_V : E_V \to V$, and an isomorphism between the covering spaces $p_U^{-1}(W) \to W$ and $p_V^{-1}(W) \to W$. We saw in the last section that this is the same as giving a covering space of B. In effect, we have therefore proved

Proposition 10.4. $\mathrm{Cov}(B) \simeq G\text{-}\mathsf{Sets}$.

By this we mean that there are constructions $\Phi : \mathrm{Cov}(B) \to G\text{-}\mathsf{Sets}$ and $\Psi : G\text{-}\mathsf{Sets} \to \mathrm{Cov}(B)$ such that

$$\Phi(\Psi(S)) \cong S$$
$$\Psi(\Phi(E)) \cong E$$

for all covering spaces $p : E \to B$ of B and all G-sets S. The definition of Φ is first to associate to a cover $p : E \to B$ a triple consisting of a covering space $p_U : p^{-1}(U) \to U$, $p_V : p^{-1}(V) \to V$, and $\alpha : p_U^{-1}(W) \cong p_V^{-1}(W)$, then to associate to these the corresponding $\pi_1(U, b)$-, $\pi_1(V, b)$-, and $\pi_1(W, b)$-sets, observe that these are compatible, and deduce that they form a G-set. The functor Ψ reverses this procedure: associate to a G-set the induced $\pi_1(U, b)$-, $\pi_1(V, b)$-, and $\pi_1(W, b)$-sets; associate to these the corresponding covers of U, V, and the isomorphism between their restrictions to W; finally, glue these together to get a cover of B.

On the other hand, $\mathrm{Cov}(B) \simeq \pi_1(B, b)\text{-}\mathsf{Sets}$ so the proposition tells us that we have an equivalence $\pi_1(B, b)\text{-}\mathsf{Sets} \simeq G\text{-}\mathsf{Sets}$. If examine exactly what this equivalence is, we will see that it associates to a $\pi_1(B, b)$-set the G-set obtained by restricting via the homomorphism $G \to \pi_1(B, b)$.

The Siefert–van Kampen theorem now follows from the following lemma:

Lemma 10.5. Suppose that $\varphi : H \to G$ is a homomorphism of groups and the induced functor $\varphi^* : G\text{-}\mathsf{Sets} \to H\text{-}\mathsf{Sets}$ is an equivalence. Then φ is an isomorphism.

Proof. We have to check that φ is a bijection. Let S denote the action of G on itself and let T denote the action of H on itself. Then we have a map of H-sets $f : T \to \varphi^*(S)$. Because φ^* is an equivalence there is some G-set S' and an isomorphism of H-sets $T \cong \varphi^*(S')$. We therefore have a H-equivariant map $f' : \varphi^*(S') \to \varphi^*(S)$. Since f will be a bijection if and only if f' is, it will be sufficient to verify that f' is a bijection.

Also because φ^* is an equivalence, there is a G-equivariant map $f'' : S' \to S$ such that $\varphi^*(f'') = f'$. Since the underlying maps of sets of the equivariant maps f' and f'' are the same, it will now be sufficient to check that f'' is a bijection.

We will show first that G acts transitively on S'. Since any equivariant map of transitive G-sets is surjective, and G obviously acts transitively on S, this will imply f'' is surjective. For this, note that $\varphi^*(S')$ is isomorphic to T, which has a transitive action of H. But this means that for every $x, y \in T$ we have some $h \in H$ with $x.h = y$. So then for every $x, y \in S'$ (recall S and T have the same underlying set) we have $x.\varphi(h) = y$. In particular, there is some element of G—namely $\varphi(h)$—carrying x to y. So the action of G on S' is transitive.

Now we verify that the map $f'' : S' \to S$ is surjective. If $y \in S$, pick any $x \in S'$. Then since g acts transitively on S there is some $g \in G$ such that $f''(x).g = y$. But then $f''(x.g) = y$ because f'' is equivariant, so y is in the image of f''. Therefore f'' is surjective.

Now we check the injectivity of f''. Suppose that $f''(x) = f''(y)$. Then since G acts transitively on S' we can write $y = x.g$ for some $g \in G$ and we get $f''(x) = f''(y) = f''(x.g) = f''(x).g$. Therefore g lies in the stabilizer of $f''(x)$. But G acts freely on S, so the stabilizer subgroup of $f''(x)$ is trivial. Therefore $g = 1$ so $y = x.g = x.1 = x$. Thus f'' is injective. □

10.3 A more general Siefert–van Kampen theorem

In fact, a fancier version of this theorem is possible. Suppose that we cover B by path connected open sets U_α such that the intersections $U_\alpha \cap U_\beta \cap U_\gamma$ are all path connected. Assume also that B is locally contractible. Let's abbreviate $U_\alpha \cap U_\beta \cap U_\gamma$ as $U_{\alpha\beta\gamma}$.

Then to give a covering space of B is the same as to give a covering space E_α of each of the U_α, along with isomorphisms $\phi_{\alpha\beta} : E_\alpha\big|_{U_\alpha \cap U_\beta} \cong E_\beta\big|_{U_\alpha \cap U_\beta}$ satisfying the **cocycle condition**—that the diagram

$$\begin{tikzcd} E_\alpha\big|_{U_{\alpha\beta\gamma}} \arrow[rr, "\phi_{\alpha\beta}\big|_{U_{\alpha\beta\gamma}}"] \arrow[dr, "\phi_{\alpha\gamma}\big|_{U_{\alpha\beta\gamma}}"'] & & E_\beta\big|_{U_{\alpha\beta\gamma}} \arrow[dl, "\phi_{\beta\gamma}\big|_{U_{\alpha\beta\gamma}}"] \\ & E_\gamma\big|_{U_{\alpha\beta\gamma}} & \end{tikzcd}$$

should commute. Then we have a map

$$G = \varinjlim_{\alpha,\beta,\gamma} \pi_1(U_{\alpha\beta\gamma}, b) \to \pi_1(B, b)$$

and

$$\mathrm{Cov}(B, F) = \mathrm{Hom}(G, \Sigma_F).$$

On the other hand, we also have

$$\mathrm{Cov}(B, F) = \mathrm{Hom}(\pi_1(B, b), \Sigma_F).$$

As we saw before a map $G \to \pi_1(B, b)$ induces an equivalence between the categories of G-sets and $\pi_1(B, b)$-sets if and only if the map $G \to \pi_1(B, b)$ is an isomorphism.

Theorem 10.6. Suppose that B is locally contractible and $B = \bigcup U_\alpha$ for open subsets U_α containing the basepoint b. Assume that all triple intersections $U_{\alpha\beta\gamma}$ are path connected. Then the natural map

$$\varinjlim_{\alpha,\beta,\gamma} \pi_1(U_{\alpha\beta\gamma}, b) \to \pi_1(B, b)$$

is an isomorphism of groups.

References

[Mun] James R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.