Math 6210 — Fall 2012
Assignment #7

Choose 5 problems to submit by Weds., Dec. 12.

For your reference, here is the definition of a semi-simplicial set.

Definition. Let $[n]$ denote the totally ordered set $\{0 < 1 < \cdots < n\}$.

A **semi-simplicial set** X is a collection of sets X_n, one for each non-negative integer n, and for each order preserving injection $u : [m] \to [m]$ a function $X_u : X_n \to X_m$ satisfying the following condition: if $[\ell] \xrightarrow{v} [m] \xrightarrow{u} [n]$ is a sequence of order preserving injections then the composition of the sequence of maps $X_n \xrightarrow{X_u} X_m \xrightarrow{X_v} X_\ell$ coincides with $X_{uv} : X_n \to X_\ell$.

If X and Y are semi-simplicial sets, then a morphism $f : X \to Y$ is a sequence of functions $f_n : X_n \to Y_n$ such that for every order preserving injection $u : [m] \to [n]$, the diagram

$$\begin{tikzcd} X_n \arrow[r, "X_u"] \arrow[d, "f_n"'] & X_m \arrow[d, "f_m"] \\ Y_n \arrow[r, "Y_u"] & Y_m. \end{tikzcd}$$

Every semi-simplicial set has a **geometric realization**, constructed in the following way. Let X be a semi-simplicial set. Begin with the discrete topological space X_0. For each $\sigma \in X_1$, attach a copy of Δ^1 going from $d_1(\sigma)$ to $d_0(\sigma)$. Then attach a 2-simplex to this space for each $\sigma \in X_2$. Inductively, for each n we get a space Y_n, with $Y_0 = X_0$, and Y_{n+1} obtained from Y_n by adjoining to Y_n a copy of Δ^{n+1} for each $\sigma \in X_{n+1}$. The attaching map for σ is determined by the $n+1$ boundary faces of $d_0(\sigma), \ldots, d_n(\sigma)$.

1 Semi-simplicial sets

Exercise 1. (a) Find a semi-simplicial model for the 2-holed torus.

(b) Find a semi-simplicial model for $\mathbf{R}\mathrm{P}^3$.

(c) Find a semi-simplicial model for S^3.

Exercise 2. [Hat, §2.1, #3] Find a semi-simplicial model for $\mathbf{R}\mathrm{P}^n$ for all n. (Hatcher has a suggestion about how to do this.)

Exercise 3. Let X and Y be semi-simplicial sets. Construct a new semi-simplicial set Z with $Z_n = X_n \times Y_n$. If $u : [m] \to [n]$ is an order preserving injection, let $Z_u = X_u \times Y_u$.

(a) Show that Z is a semi-simplicial set.

(b) Show with an example that $|Z| \neq |X| \times |Y|$.

This defect is one reason topologists prefer simplicial sets to semi-simplicial sets.

Exercise 4. Let X be a topological space. Define operators $P_i : C_n(X, \mathbf{Z}) \to C_{n+1}(X \times I, \mathbf{Z})$ by the following rule. Given $\sigma : \Delta^n \to X$, let $\sigma' : \Delta^n \times I \to X \times I$ be the induced map. Let $v_i \in \Delta^n \times I$ be the point with coordinates $(e_i, 0)$ and let $w_i \in \Delta^n \times I$ be the point with coordinates $(e_i, 1)$ (here we are viewing Δ^n as a subset of $\mathbf{R}^{n+1}$ by way of its barycentric coordinates and e_i is the i-th standard basis vector of $\mathbf{R}^{n+1}$). For points $a_1, \ldots, a_k \in \Delta^n \times I$ that are not contained in a $(k-1)$-dimensional plane, let $[a_1, \ldots, a_k]$ be the simplex they span. With this notation, [1] $\leftarrow_1$

$$P_i(\sigma) = \sigma\big|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}.$$

Define $P(\sigma) = \sum_{i=0}^{n} (-1)^i P_i(\sigma)$ and extend by linearity to get a map $P : C_n(X, \mathbf{Z}) \to C_{n+1}(X \times I, \mathbf{Z})$.

Verify the following formula that was stated in class:

$$\partial P(\sigma) - P(\partial \sigma) = \sigma \times \{1\} - \sigma \times \{0\}.$$

Exercise 5. Let $f, g : X \to Y$ be homotopic maps. Show that f^* and g^* give the same map $H^n(Y, \mathbf{Z}) \to H^n(X, \mathbf{Z})$.

2 Eilenberg–Mac Lane spaces

In this section, we will show that a $K(G, 1)$ exists for each group G.

Exercise 6. A labelling of the edges of Δ^n by elements of G is a function λ from the set of edges of Δ^n to the set G. To give such a labelling, we give a value $\lambda(i, j) \in G$ for every $i \leq j$. We define BG_n to be the set of all ways of labelling the edges of Δ^n by elements of G satisfying the following property: $\lambda(i, j)\lambda(j, k) = \lambda(i, k)$ (the product here is the group operation) for all $i \leq j \leq k$ in the set $\{0, 1, \ldots, n\}$.

Note that if $u : [m] \to [n]$ is a monotonic injection corresponding to a face of Δ^n then the composition of λ with u is a labelling of the edges of Δ^m by G satisfying the compatibility condition explained above.

(a) (optional) Let Δ^n denote the category whose objects are the integers $0, 1, \ldots, n$ and in which $\mathrm{Hom}(i, j)$ is empty for $j < i$ and consists of exactly one morphism for $j \geq i$. Let BG denote the category with one object, $*$, and $\mathrm{Hom}(*, *) = G$; the rule for composition is the group law in G. Verify that BG_n, as defined above, is the set of functors from Δ^n to BG.

(b) Verify that with the definitions above, BG is a semi-simplicial set.

[1] correction: originally this said $P_i(\sigma) = [v_0, \ldots, v_i, w_i, \ldots, w_n]$; thanks Jonathan Lamar

(c) Compute $\pi_1(\lvert BG \rvert)$. (Hint: use the Siefert–van Kampen theorem.)

By definition, the **group homology** of G (with coefficients in $\mathbf{Z}$) is $H_*(BG, \mathbf{Z})$; the **group cohomology** of G (with coefficients in $\mathbf{Z}$) is $H^*(BG, \mathbf{Z})$.

Exercise 7. Let G be a group. Define EG_n to be the set of labellings of the vertices of Δ^n by elements of G. An element of EG_n is thus a function $\mu : [n] \to G$.

If $u : [m] \to [n]$ is an order preserving function corresponding to a m-dimensional face of Δ^n, let $EG_u(\mu)$ be the labelling of the vertices of Δ^m corresponding to the function $\mu \circ u$.

(a) (optional) Let C be the category whose objects are the elements of G and in which there is exactly one morphism between any two objects. Show that EG_n is the set of functors from the category Δ^n (as defined in the last exercise) to C.

(b) Verify that with this definition, EG is a semi-simplicial set.

(c) Show that $\lvert EG \rvert$ is contractible.

Exercise 8. Let μ be a labelling of the vertices of Δ^n by elements of G. Define a labelling of the edges of Δ^n by the rule

$$\lambda(i,j) = \mu(j)\mu(i)^{-1}.$$

(a) Show that this defines a map of semi-simplicial sets $p : EG_n \to BG_n$. Conclude that we obtain a continuous map of topological spaces $\lvert p \rvert : \lvert EG \rvert \to \lvert BG \rvert$.

(b) [Hat, §1.B, #1] Show that this map makes $\lvert EG \rvert$ into a covering space of $\lvert BG \rvert$.

(c) Deduce (using the previous exercise) that $\pi_n(\lvert BG \rvert) = 0$ for all $n \geq 2$. Conclude that $\lvert BG \rvert$ is a $K(G, 1)$.

3 Homology and cohomology

In this section, you are asked to compute homology and cohomology with coefficients in a general commutative ring A. If you aren't comfortable doing that then do separate computations for $A = \mathbf{Z}$, $A = \mathbf{Q}$, and $A = \mathbf{F}_2$. (And if you aren't comfortable doing that then just do the computation for $A = \mathbf{Z}$.)

Exercise 9. Let X be the oriented surface of genus 2 (the 2-holed torus). Compute the homology and cohomology of X with coefficients in a commutative ring A.

Exercise 10. Compute the homology and cohomology of $\mathbf{R}\mathrm{P}^3$ with coefficients in a commutative ring A.

Exercise 11. Compute the homology and cohomology of S^3 with coefficients in a commutative rings A.

Exercise 12. [Hat, §2.1, #8]

Exercise 13. [Hat, §2.1, #9]

Exercise 14. Suppose that X is a retract of Y.

(a) [Hat, §2.1, #11] Show that $H_n(X, A) \to H_n(Y, A)$ is injective.

(b) Show that $H^n(Y, A) \to H^n(X, A)$ is surjective.

References

[Hat] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.