Math 6210 — Fall 2012
Assignment #6

Choose 5 problems to submit by Fri., Nov. 30. Remember to cite your soources.

1 Fundamental groups of graphs and an application to group theory

Exercise 1. (a) Let $X = S^1 \vee \cdots \vee S^1$ be a finite wedge of n circles. Demonstrate that $\pi_1(X, *)$ is a free group generated by the maps $S^1 \to X$ corresponding to the inclusions of the n circles.

(b) Suppose that X is a wedge of n circles and Y is a wedge of m circles. Show that $X \simeq Y$ if and only if $n = m$.

Exercise 2. Suppose that $X_1 \subset X_2 \subset \cdots$ is an increasing union of subspaces of a topological space X. Assume that all X_i contain the basepoint x of X.

(a) Construct a homomorphism $\varinjlim \pi_n(X_i, x) \to \pi_n(X, x)$ for each n.

(b) Assume that the X_i are all open in X and their union is X.[1] Prove that the map constructed in the last section is an isomorphism. $\leftarrow_1$

(c) Let $X = \bigvee_{\alpha \in A} S^1$ be a wedge of circles indexed by a set A. Prove that $\pi_1(X, *)$ is a free group generated by the maps $S^1 \to X$ corresponding to the elements of A.

Exercise 3. Let G be a graph. We allow G to have edges connecting vertices to themselves. Let V be the set of vertices of G and E the set of edges. Define a topological space $X(G)$: Begin with the set V as a discrete topological space. For each edge e in E, adjoin to V a copy of $[0, 1]$ connecting the two vertices to which e is incident.

(a) Prove that $X(G)$ is homotopy equivalent to a wedge of circles.

(b) Conclude that $\pi_1(X(G), v)$ is a free group for any graph G and any vertex v of G.

(c) Show that any covering space of $X(G)$ is homeomorphic to $X(G')$ for some graph G'.

(d) Conclude that any subgroup of a free group is a free group.

[1] originally forgot to include the assumption that the union was X

2 Eilenberg–Mac Lane spaces

A based topological space (X, x) is said to be a $K(G, 1)$ if X is path connected, $\pi_1(X, x) \cong G$, and $\pi_n(X, x) = 0$ for all $n \geq 2$.

Exercise 4. Let $p : E \to B$ be a covering space and b a point of B. Prove that the map

$$p_* : \pi_n(E, e) \to \pi_n(B, b)$$

is an isomorphism for all $n \geq 2$.

Exercise 5. (a) Prove that S^1 is a $K(\mathbf{Z}, 1)$.

(b) Prove that if X_i is a $K(G_i, 1)$ for all i in some set I then $\prod_{i \in I} X_i$ is a $K(\prod_{i \in I} G_i, 1)$.

(c) Conclude that the n-torus is a $K(\mathbf{Z}^n, 1)$.

Exercise 6. Let S^∞ be the set of sequences $x_1, x_2, \ldots$ of real numbers such that

(i) all but finitely many x_i are zero, and

(ii) $\sum x_i^2 = 1$.

Let $\mathbf{R}\mathrm{P}^\infty = S^\infty / \{\pm 1\}$.

(a) Prove that S^∞ is contractible.

(b) Prove that $\mathbf{R}\mathrm{P}^\infty$ is a $K(\mathbf{Z}/2\mathbf{Z}, 1)$.

3 Siefert–van Kampen

Exercise 7. Let $(X, *)$ and $(Y, *)$ be locally contractible topological spaces. Prove that $\pi_1(X \vee Y, *) \cong \pi_1(X) * \pi_1(Y)$.

Exercise 8. [Hat, §1.2, # 10] Let X be the cube $[-2, 2] \times [-2, 2] \times [-2, 2]$. Let[2] $\leftarrow_2$

$$\begin{aligned} A &= X \cap \Big\{ (x, y, 0) \;\Big|\; \frac{1}{9}(x+2)^2 + \frac{1}{4}y^2 = 1 \Big\} \\ B &= X \cap \Big\{ (x, 0, z) \;\Big|\; \frac{1}{9}(x-2)^2 + \frac{1}{4}z^2 = 1 \Big\} \\ C &= \Big\{ (x, y, z) \;\Big|\; y^2 + z^2 = 4 \Big\} \subset X \end{aligned}$$

Let $Y = X \smallsetminus (A \cup B)$.

(a) Compute $\pi_1(Y)$.

(b) Show that C represents a non-zero element of $\pi_1(Y)$.

[2] correction in the equations below; thanks Megan

4 Covering spaces

Exercise 9. Let $p : E \to B$ and $p' : E' \to B$ be two covering spaces. Assume that B is path connected and locally contractible and let b be a basepoint of B. Let S and S' be the two $\pi_1(B, b)$-sets corresponding to E and E'. We can form a new $\pi_1(B, b)$-set by taking the product $S \times S'$. Prove that the corresponding covering space of B is $E \times_B E'$.

Exercise 10. [Hat, §1.3, #14] Describe all connected covering spaces of $\mathbf{R}\mathrm{P}^2 \vee \mathbf{R}\mathrm{P}^2$.

References

[Hat] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.