Math 6210 — Fall 2012
Assignment #5

Choose 5 problems to submit by Weds., Nov. 14. Complete at least one problem from each section. Remember to cite your soources.

Thoughout this problem set, assume that B is path connected and locally contractible.

1 The homotopy exact sequence

Exercise 1. Let $p : (E, e) \to (B, b)$ is a covering space and let $F = p^{-1}(b)$. Prove that the sequence

$$\pi_1(F, e) \to \pi_1(E, e) \to \pi_1(B, b) \to \pi_0(F, e) \to \pi_0(E, e) \to \pi_0(B, b) \to 1$$

is exact. You should check the following things:

(a) $\pi_1(F, e)$ is the trivial group,

(b) $\pi_1(E, e) \to \pi_1(B, b)$ is injective,

(c) the image of $\pi_1(E, e) \to \pi_1(B, b)$ is the stabilizer of e when $\pi_1(B, b)$ acts on $F = \pi_0(F, e)$,

(d) the quotient of $F = \pi_0(F, e)$ by the action of $\pi_1(B, b)$ is the set of elements of $\pi_0(E, e)$ that map to $b \in \pi_0(B, b)$, and

(e) $\pi_0(E, e) \to \pi_0(B, b)$ is surjective.

We will prove a much more general version of this sequence later.

2 Applications

Exercise 2. Prove that there is no continuous retraction from the disc $D^2 = \{(x, y) \mid x^2 + y^2 \le 1\}$ onto the circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$.

Exercise 3. Prove the Brouwer fixed point theorem for this disc D^2: If $f : D^2 \to D^2$ is a continuous map then there is some $x \in D^2$ such that $f(x) = x$. (Hint: consider the map $g(x) = \frac{f(x)-x}{|f(x)-x|}$.)

3 Fundamental groups of surfaces

Exercise 4. For each $n \ge 0$, compute $\pi_1(T)$ where T is the n-holed torus.

4 Cetera

Exercise 5. Use the following steps to show that $\pi_m(S^n, *) = 0$ for $m < n$.

(a) Show that if $f : (S^m, *) \to (S^n, *)$ is homotopic to a map that is not surjective then it is homotopic to a constant map.

(b) Argue that it is possible to break I^m up into **simplices** of the form

$$X = \Big\{\sum_{i=0}^{m} a_i x_i \;\Big|\; \forall i, a_i \geq 0 \text{ and } \sum a_i = 1\Big\}$$

such that the simplices intersect along their faces and satisfy the following property: if $p_0, \ldots, p_{m+1} \in f(X)$ then the hyperplane in $\mathbf{R}^{n+1}$ that contains all of the p_i does not pass through 0. (Hint: it might be helpful to look up **barycentric subdivision**.) Where does the hypothesis $m < n$ come in?

(c) For each simplex X in your partition of I^m, let

$$g_X\big(\sum a_i x_i\big) = \sum a_i f(x_i).$$

Then let $h_X(y) = \frac{g_X(y)}{|g_X(y)|}$. Show that h_X is well-defined and continuous.

(d) Show that if two simplices X and X' intersect along a face $X \cap X'$ then $h_X\big|_{X\cap X'} = h_{X'}\big|_{X\cap X'}$. Conclude the the functions h_X glue together to give a continuous function $h : (I^m, \partial I^m) \to (S^n, *)$.

(e) Construct a homotopy between f and h.

(f) Show that h is not surjective.

(g) Conclude that $\pi_m(S^n, *) = 0$ for $m < n$.

Exercise 6. Let G be a group. Show that

(a) there is a G-set S such that for any G-set F there is a bijection $\alpha_F : \mathrm{Hom}_G(S, F) \cong F$ and for any G-equivariant function $\varphi : F \to F'$ the diagram

$$\begin{tikzcd} \mathrm{Hom}_G(S,F) \arrow[r, "\alpha_F"] \arrow[d, "\mathrm{Hom}_G(S,\varphi)"'] & F \arrow[d, "\varphi"] \\ \mathrm{Hom}_G(S,F') \arrow[r, "\alpha_{F'}"] & F' \end{tikzcd}$$

commutes, and

(b) if S' is any other G-set and $\alpha' : \mathrm{Hom}_G(S', F)$ another system of maps with the same property then there is a unique isomorphism of G-sets $\varphi : S \to S'$ such that the diagram

$$\begin{tikzcd} \mathrm{Hom}_G(S,F) \arrow[r, "\alpha_F"] & F \\ \mathrm{Hom}_G(S',F) \arrow[u, "\mathrm{Hom}_G(\varphi,F)"] \arrow[ur, "\alpha'_F"'] & \end{tikzcd}$$

commutes.

If you do this exercise successfully, you will have done all the work necessary to prove the Yoneda lemma. In fact, if you are comfortable with the categorical language, you might just want to prove the Yoneda lemma and apply it here.

Exercise 7. This problem assumes some knowledge about category theory. It has several important applications in algebraic geometry.

Let G be a group and let $\mathscr{C}$ be the category of right G-sets. Define a functor $F : \mathscr{C} \to \mathsf{Sets}$ that forgets the G-action (it sends a G-set S to the underlying set of the action). Construct a canonical isomorphism of groups $\mathrm{Aut}(F) \cong G$.

Exercise 8. Let B be a topological space and let X be the subset of $\mathrm{Cont}(I, B) \times \mathrm{Cont}(I, B)$ consisting of those pairs (f, g) such that $f(1) = g(0)$. The map $\mu : X \to \mathrm{Cont}(I, B)$ defined by

$$\mu(f,g)(t) = \begin{cases} f(2t) & t \le \frac{1}{2} \\ g(2t-1) & t \ge \frac{1}{2} \end{cases}$$

is continuous.

Exercise 9. Let $p : (E, e) \to (B, b)$ be a covering space. Show that the induced map $\mathrm{Cont}((I, 0), (E, e)) \to \Omega(B, b)$ is a *homeomorphism*. (We have already seen that this is a bijection, so you don't have to prove that.)

Exercise 10. (a) Let $G = \pi_1(B, b)$ and suppose that G acts on sets S and S'. Let E and E' be the associated covering spaces of B. The group G also acts on $S \amalg S'$ in a natural way. Prove that covering space of B associated to this action is $E \amalg E'$.

(b) Let $p : E \to B$ be a covering space. Prove that the orbits of $\pi_1(B, b)$ acting on $p^{-1}(b)$ are in bijection with the path components of E. (Hint: the homotopy exact sequence might be helpful.)

Exercise 11. The Hawai'ian earring is the topological space $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \subset \mathbf{R}^2$ is the subspace

$$\big\{(x,y) \bigm| (x - \frac{n-1}{n})^2 + y^2 = \frac{1}{n^2}\big\}.$$

Choose $p = (1, 0)$ as the basepoint.

(a) Show that the fundamental group of the Hawai'ian earring is not abelian.[1] $\leftarrow_1$

(b) Show that the Hawai'ian earring does not have a simply connected covering space.

(c) Show that the quotient topology on $\pi_1(X, p)$ from the surjective function $\Omega(X, p) \to \pi_1(X, p)$ is not discrete.

Exercise 12. [Hat, §1.1, #17] Find infinitely many pairwise non-homotopic retractions $S^1 \vee S^1 \to S^1$.

Exercise 13. Let $f : X \to Z$ and $g : Y \to Z$ be continuous maps. Let $X \times_Z Y$ be the fiber product of X and Y over Z and let $p : X \times_Z Y \to X$ and $q : X \times_Z Y \to Y$ be the two projection maps. Prove the universal property of $X \times_Z Y$: if $a : W \to X$ and $b : W \to Y$ are continuous maps such that $fa = gb$ then there is a unique continuous map $c : W \to X \times_Z Y$ such that $pc = a$ and $qc = b$.

Diagrammatically, you are supposed to show that given a commutative diagram of solid lines

$$\begin{tikzcd}
W \arrow[dr, dashed, "c"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & \\
& X \times_Z Y \arrow[r, "p"'] \arrow[d, "q"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

there is a unique dashed arrow making the whole diagram commute.

Exercise 14. (a) Check that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of continuous maps and $W \to Z$ is another continuous map then there is a canonical homeomorphism

$$X \underset{Y}{\times} (Y \underset{Z}{\times} W) \cong X \underset{Z}{\times} W.$$

(Hint: the universal property may help.)

(b) Suppose that X is a subspace of Z and $f : Y \to Z$ is a continuous map. Verify that $X \times_Z Y \cong f^{-1}(X)$.

(c) Suppose that $f : X \to Z$ and $g : Y \to Z$ are continuous maps. Let $p : X \times_Z Y \to X$ and $q : X \times_Z Y \to Y$ be the two projections. Show that for each $x \in X$ the map

$$g\big|_{p^{-1}(x)} : p^{-1}(x) \to g^{-1}(f(x))$$

is a homeomorphism.

Exercise 15. Verify that if $p : E \to Y$ is a covering space and $f : X \to Y$ is a continuous map then the projection $q : E \times_Y X \to X$ is a covering space.

[1] Replaced the original problem with this easier one.

References

[Hat] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.