Math 6210 — Fall 2012
Assignment #4

Choose 5 problems to submit by Weds., Oct. 17 with at least one problem from each section. Remember to cite your sources.

Exercise 1. Suppose that Y is discrete. Let X be a topological space and let X^Y be the set of functions from Y to X. Show that the product topology on X^Y coincides with the compact open topology.

Exercise 2. A **pointed topological space** is a pair (X, x) where X is a topological space and x is a point of X. Usually we leave the point tacit and refer to X as the pointed topological space. If X and Y are based topological spaces, let $\mathrm{Cont}_*(X, Y)$ be the subspace of $\mathrm{Cont}(X, Y)$ consisting of those $f : X \to Y$ that preserve basepoints—that is, those $f : X \to Y$ such that $f(x) = y$. Here are two constructions concerning pointed topological spaces:

(i) First choose a basepoint for S^1. The **based loop space** of a pointed topological space is $\Omega X := \mathrm{Cont}_*(S^1, X)$. Notice that ΩX is a pointed topological space: the basepoint is the constant map sending all of S^1 to the basepoint $x \in X$.

(ii) The **smash product** of pointed topological spaces (X, x) and (Y, y) is denoted $X \wedge Y$ and is obtained by first forming the product $X \times Y$ and then collapsing the subspace $X \times \{y\} \cup \{x\} \times Y$ to a point. The result is given the quotient topology. Notice that it has a basepoint, the point to which $X \times \{y\} \cup \{x\} \times Y$ was collapsed.

(iii) The **suspension** of a pointed topological space (X, x) is the smash product with a circle. It is denoted $\Sigma X = S^1 \wedge X$.[1] ←1

These constructions are of fundamental importance in homotopy theory.

Let (X, x) and (Y, y) be based topological spaces. Construct a natural bijection

$$\mathrm{Cont}_*(Y, \Omega X) = \mathrm{Cont}_*(\Sigma Y, X).$$

Exercise 3. (a) Let (X, x) and (Y, y) be pointed spaces. Construct a canonical map $\pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y)$ and show it is an isomorphism of groups.

(b) Compute $\pi_1(T, t)$ where T is the torus and t is any basepoint.

Exercise 4. (a) Show that $\mathbf{R}^2 \smallsetminus \{0\}$ is homotopy equivalent to S^1.

[1] forgot to include the definition of ΣX originally; thanks Megan and Tom

(b) Compute $\pi_1(\mathbf{R}^2 \smallsetminus \{0\}, x)$ where x is any basepoint. (Hint: use Exercise 12.)

Exercise 5. (a) Prove that $\mathbf{R}^1 \cong \mathbf{R}^n$ if and only if $n = 1$.

(b) Prove that $\mathbf{R}^2 \cong \mathbf{R}^n$ if and only if $n = 2$.

Exercise 6. [Hat, §1.1, #5] Let X be a topological space. Show that the following three conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map (with image a point).

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) For all $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.

Exercise 7. [Hat, §1.1, #6] Let $[S^1, X]$ denote the set of homotopy classes of maps from the circle to X (not preserving basepoints). There is a natural map $\Phi : \pi_1(X, x) \to [S^1, X]$ by forgetting basepoints. Show that this identifies $[S^1, X]$ with the set of conjugacy classes in $\pi_1(X, x)$.

Exercise 8. (a) Prove that $\Sigma S^n \cong S^{n+1}$. (Hint: Use the fact that S^n can be obtained by collapsing the boundary of $[0, 1]^n$ to a point.)

(b) Use this and the fact that for pointed spaces X and Y we have $\mathrm{Cont}_*(\Sigma X, Y) = \mathrm{Cont}_*(X, \Omega Y)$ to demonstrate that $\pi_n(X, x)$ is a group for all pointed spaces (X, x) and all $n > 0$. (Hint: Show that $\pi_n(X) = \pi_{n-1}(\Omega X)$ for $n > 0$.)

Proof. We know that $\mathrm{Cont}_*(S^n, X) = \mathrm{Cont}_*(S^{n-1}, \Omega X)$ since $\Sigma S^{n-1} = S_n$. We have to check that homotopy gives the same equivalence relation under this identification. The set of based homotopies between maps in $\mathrm{Cont}_*(S^n, X)$ is $\mathrm{Cont}((I \times S^n, I \times *), (X, *))$. We can identify this with $\mathrm{Cont}((I \times S^1 \times S^{n-1}, I \times S^1 \times * \cup I \times * \times S^{n-1}), (X, *))$. But this may in turn be identified with $\mathrm{Cont}((I \times S^{n-1}, I \times *), (\mathrm{Cont}((S^1, *), (X, x)), *)) = \mathrm{Cont}((I \times S^{n-1}, I \times *), \Omega X)$, which is precisely the set of based homotopies in ΩX. Therefore homotopy imposes the same equivalence relation and $[(S^n, *), (X, *)] = [(S^{n-1}, *), (\Omega X, *)]$. In other words, $\pi_n(X) = \pi_{n-1}(\Omega X)$.

Now, by induction, we can identify $\pi_n(X) = \pi_1(\Omega^{n-1} X)$ so $\pi_n(X)$ gets a group structure from the group structure on $\pi_1(\Omega^{n-1} X)$. $\square$

Exercise 9. Let $f, g : X \to Y$ be continuous functions. Suppose that $H : f \simeq g$ and $H' : f \simeq g$ are two homotopies from f to g. A **homotopy between homotopies** is a continuous function

$$\varphi : [0, 1] \times [0, 1] \times X \to Y$$

such that

$$\varphi\big|_{\{0\}\times[0,1]\times X} = H \qquad\qquad \varphi\big|_{\{1\}\times[0,1]\times X} = H'$$

and $\varphi\big|_{[0,1]\times\{0\}\times X}$ is the constant homotopy from f to itself and $\varphi\big|_{[0,1]\times\{1\}\times X}$ is the constant homotopy from g to itself.

(a) Let $H : [0,1]\times X \to Y$ be a homotopy from f to g and let $H' : [0,1]\times Y \to Y$ be a homotopy from g to h. Verify that

$$F(t,x) = \begin{cases} H(2t,x) & t \le 1/2 \\ H(2t-1,x) & t \ge 1/2 \end{cases}$$

defines a homotopy from f to h. Write $F = H' \circ H$ for this homotopy.

(b) Show that this composition law is almost never associative

(c) Verify that this composition law is **homotopy associative** in the follwing sense: the homotopies $H \circ (H' \circ H'')$ and $(H \circ H') \circ H''$ are homotopic to one another.

(d) Use this to prove that $\pi_1(X,x)$ is a group. (Hint: view $\pi_1(X,x)$ as the set of homotopies from the map $x : (\text{point}) \to X$ to itself.)

Exercise 10 (Cf. [Mun, §52, #7])**.** A **group object in groups** is a group G equipped with *two* group structures $m : G\times G \to G$ and $m' : G\times G \to G$ such that each map m and m' is a homomorphism with respect to the other group structure. Note that each of these is associative, has an identity element, and has inverses, but the identity elements and inverses need not be the same (at least a priori).[2] ←2

(a) Suppose that G is a group object in groups. Show that $m = m'$ and the group structure is abelian. (Hint: first check that $m(m'(x,y), m'(z,w)) = m'(m(x,z), m(y,w))$ for all $x,y,z,w \in G$; then check that the identity elements in the two group structures are the same.)

(b) Let X be a based topological space. Construct a continuous map $\Omega X \times \Omega X \to \Omega X$ that concatenates loops.

(c) Show that for any space Y, this induces a function $[Y,\Omega X]\times[Y,\Omega X] \to [Y,\Omega X]$. Verify that this gives $[Y,\Omega X]$ a group structure.

(d) Deduce that $[S^1,\Omega X]$ has **two** group structures: one coming from the usual group structure on $\pi_1(\Omega X)$ and the other coming from the group structure constructed above.

[2] definition changed here (the definition that was here earlier is equivalent but is harder to use in this problem); also added a hint or two below.

(e) Let m and m' denote the multiplication functions for these two group structures. Show that each one is a homomorphism with respect to the other.

(f) Conclude that for any based space X, the set $\pi_2(X)$ has the structure of a group object in groups. Deduce that $\pi_2(X)$ is abelian for any based space X.

(g) Show $\pi_n(X)$ is abelian for all $n \geq 2$ and any based space X. (Hint: use the fact that $\pi_n(X) = \pi_{n-1}(\Omega X)$.)

Exercise 11. Let (X, x), (Y, y), and (Z, z) be pointed spaces. Assume that $f : (Y, y) \to (Z, z)$ is a continuous map. This induces a function

$$f_* : \mathrm{Cont}_*((X, x), (Y, y)) \to \mathrm{Cont}_*((X, x), (Z, z))$$

defined by $f_*(g) = f \circ g$.

(a) Show that this induces a well-defined function

$$f_* : [(X, x), (Y, y)] \to [(X, x), (Z, z)].$$

(b) Deduce that there is an induced map $f_* : \pi_n(Y, y) \to \pi_n(Z, z)$ for all n.

(c) Check that the map $f_* : \pi_1(Y, y) \to \pi_1(Z, z)$ is a homomorphism.

Exercise 12. Suppose that (X, x) and (Y, y) are based spaces. A based homotopy equivalence between (X, x) and (Y, y) is a continuous map $f : (X, x) \to (Y, y)$ such that there is a continuous function $g : (Y, y) \to (X, x)$ and based homotopies between $gf : (X, x) \to (X, x)$ and id_X and between $fg : (Y, y) \to (Y, y)$ and id_Y.

Show that if $f : (X, x) \to (Y, y)$ is a based homotopy equivalence then the induced map $\pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism for all n. (Hint: it might be helpful to use the previous exercise.)

References

[Hat] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Mun] James R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.