Math 6210 — Fall 2012

Assignment #3

Choose 8 problems to submit by Weds., Oct. 3 with at least one problem from each section. Remember to cite your sources.

1 Compact spaces and Hausdorff spaces

Exercise 1. Show that if $f : X \to Y$ is a continuous surjection between topological spaces and X is compact then Y is compact.

Exercise 2. Prove that the following spaces are all compact:

(a) $[0, 1] \times [0, 1]$,

(b) the ball $B(0, 1) \subset \mathbf{R}^n$,

(c) S^n,

(d) $\mathbf{R}P^n$,

(e) the Klein bottle,

(f) the n-holed torus.

Exercise 3. Let X be a Hausdorff topological space.

(a) Let $Z, W \subset X$ be disjoint compact subsets. Show that there exists disjoint open subsets $U, V \subset X$ such that $Z \subset U$ and $W \subset V$. (Hint: consider the case $Z = \{z\}$ first.)

(b) Show if X is Hausdorff and $x \in X$ is a point then $\{x\} \subset X$ is closed.

(c) Let X be a Hausdorff topological space. Show that a compact subset of X is closed in X.

Exercise 4. (a) Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \big| x \in X\} \subset X \times X$ is closed.

(b) Suppose that f and g are two continuous maps from a topological space X to a Hausdorff space Y. Let $Z \subset X$ be the subspace consisting of all points $x \in X$ such that $f(x) = g(x)$. Show that Z is closed in X. (Hint: consider the map $(f, g) : X \to Y \times Y$.)

2 Universal properties

Exercise 5. [Mun, §22, #2(b)] Suppose that X is a topological space and $A \subset X$ is a subset. Let $r : X \to X$ be a continuous function whose image is contained in A and that restricts to the identity on A. Show that the subspace topology on A (from the inclusion $A \subset X$) and the quotient topology (from the map $r : X \to A$) are the same.

Exercise 6. Let I be a partially ordered set. Assume that for each $\alpha \in I$ we have a topological space X_α and that whenever $\alpha < \beta$ in I we have a continuous map $f_{\alpha\beta} : X_\alpha \to X_\beta$. If $\alpha < \beta < \gamma$ then the diagram

$$\begin{tikzcd} X_\alpha \arrow[rr, "f_{\alpha\gamma}"] \arrow[dr, "f_{\alpha\beta}"'] & & X_\gamma \\ & X_\beta \arrow[ur, "f_{\beta\gamma}"'] & \end{tikzcd}$$

commutes. That is, $f_\alpha\gamma = f_{\beta\gamma} \circ f_{\alpha\beta}$. An **inverse limit** (or **projective limit**, or just **limit**) of $\{X_\alpha\}$ is the universal (final) example of a topological space X and continuous maps $p_\alpha : X \to X_\alpha$ for every $\alpha \in I$ such that, whenever $\alpha < \beta$ in I, the diagram

$$\begin{tikzcd} & X \arrow[dl, "p_\alpha"'] \arrow[dr, "p_\beta"] & \\ X_\alpha \arrow[rr, "f_{\alpha\beta}"] & & X_\beta \end{tikzcd}$$

commutes. That is $f_\alpha\beta \circ p_\alpha = p_\beta$. We denote the limit of the X_α by $\lim_{\alpha\in I} X_\alpha$ or by $\varprojlim_{\alpha\in I} X_\alpha$.

(a) Suppose that $X = \prod_{\alpha\in I} X_\alpha$. Verify that if I is given the trivial partial order, in which $\alpha \leq \beta$ if and only if $\alpha = \beta$, then $X = \varprojlim_{\alpha\in I} X_\alpha$. (Hint: use the universal property.)

(b) Show that if $X = \varprojlim_{\alpha\in I} X_\alpha$ and $Y = \prod_{i\in I} X_\alpha$ there is a continuous map $i : X \to Y$ such that $q_\alpha \circ i = p_\alpha$.

(c) Show that i is injective by considering all continuous functions from a point to $\varprojlim_{\alpha\in I} X_\alpha$.

(d) Let $Y = \prod_{i\in I} X_\alpha$ and let Z be the subset of all $y \in Y$ such that $f_{\alpha\beta} \circ p_\alpha(y) = p_\beta(y)$. Show that if Z is given the subspace topology then it has the universal property of the inverse limit.

(e) Show that if the X_α are Hausdorff for every α then $\varprojlim X_\alpha$ is Hausdorff.

(f) Show that if the X_α are compact and Hausdorff for every α then $\varprojlim X_\alpha$ is compact.

(g) Prove that $\mathbf{Z}_p = \varprojlim_n \mathbf{Z}/p^n\mathbf{Z}$.

3 Connectedness

Exercise 7. (a) Suppose that X and Y are connected subspaces of a topological space Z and that $X \cap Y \neq \varnothing$. Show that $X \cup Y$ is connected.

(b) Let X be a topological space and suppose that for every pair of points $x, y \in X$ there is a connected subspace $Y \subset X$ with $x, y \in Y$. Show that X is connected.

(c) Suppose $X \subset Y$ and X is connected. Show that $\overline{X}$ is connected.

(d) Declare that $x \sim y$ if there is a connected subset of X that contains both x and y. Show that $\sim$ is an equivalence relation. The equivalence classes under this relation are called the **components** of X.

(e) Show that the components of any topological space are connected and closed.

(f) Give an example of a topological space whose components are not open.

Exercise 8. (a) Show that if X and Y are connected topological spaces, so is $X \times Y$. (Hint: use the previous exercise.)

(b) Show that if I is finite and each X_i is a connected topological space then $\prod_{i \in I} X_i$ is connected.

(c) Suppose that I is an indexing set and for each i we have a connected topological space X_i. Prove that $X = \prod_{i \in I} X_i$ is connected.

Exercise 9. (a) Prove that $[0, 1]$ is connected.

(b) Prove that $\mathbf{R}$ is connected.

(c) Prove that $\mathbf{R}^n$ is connected.

4 Problems related to Tychonoff's theorem

Exercise 10. In this exercise we will see that a topological space can be compact without being sequentially compact. That is, a compact topological space can have sequences with no convergent subsequences.

(a) Let $X = \prod_{i \in I} X_i$. Let $p_i : X \to X_i$ be the projection. Suppose that $x_1, x_2, \ldots$ is a sequence in X. Show that $x_1, x_2, \ldots$ converges if and only if the sequenece $p_i(x_1), p_i(x_2), \ldots$ converges in X_i for each i.

(b) Let 2 be the set $\{0, 1\}$ with the discrete topology. Let $X = 2^{\mathbf{N}} = \prod_{i=0}^{\infty} 2$. Show that X is compact.

(c) Show that a sequence $x_1, x_2, \ldots$ in X converges to y if and only if every term is eventually constant.

(d) Let $Y = 2^X = \prod_{x \in X} \{0,1\}$. Show that Y is compact.

(e) We view X as the set of sequences drawn from the set $\{0,1\}$ and we view Y as the set of functions from X to $\{0,1\}$. Define a sequence in Y by $f_n(x) = x_n$. Show that this sequence has no convergent subsequence.

Exercise 11. Recall from the first problem set that a topological space X is said to have a **sequential topology** if closed subsets of X are exactly those subsets $Z \subset X$ that contain all limits of sequences drawn from Z.

(a) Show that metric topologies are sequential.

(b) A topological space is said to be **sequentially compact** if every sequence contains a convergent subsequence. Show that a sequential space is compact if and only if it is sequentially compact.

(c) Show that a countable product of sequentially compact spaces is sequentially compact.

Exercise 12. Prove, using ultrafilters, that an arbitrary product of Hausdorff topological spaces is Hausdorff.

Exercise 13. Here we will see that the axiom of choice and Tychonoff's theorem are equivalent. Since we have already seen that the axiom of choice implies Tychonoff's theorem, we only have to prove the converse here.

(a) Let X_i, $i \in I$ be a collection of non-empty sets. For each i, give X_i the indiscrete topology and let $X'_i = X_i \amalg \{\infty_i\}$. Show that $X' = \prod_{i \in I} X'_i$ is compact.

(b) Let $p_i : X' \to X'_i$ be the projection. Show that the sets $p_i^{-1}(\infty_i)$ are open.[1] ←1

(c) Show that there is no finite subset $J \subset I$ such that $\bigcup_{i \in I} p_i^{-1}(\infty_i) = \bigcup_{i \in J} p_i^{-1}(\infty_i)$.[2] ←2

(d) Show that the sets $p_i^{-1}(\infty_i)$ cover X' if and only if $\prod_{i \in I} X_i = \varnothing$.[3][4] ←3 ←4

Exercise 14. Let F be a collection of subsets of a set X. Show that F is an ultrafilter if and only if it satisfies the following beautiful condition:[5] ←5

> If X is the disjoint union of subsets $A_1, \ldots, A_n$ then there is exactly one i such that $A_i \in F$.

[1] Correction! This erroneously said $p_i^{-1}(X_i)$ before; the problem wasn't asking you to prove something false in that case, but it's unfortunatley not useful for proving the axiom of choice. Thanks to Megan for noticing this!

[2] same correction as in the last part

[3] Correction! before this said $X = \varnothing$ without defining what X was; thanks Nikki and Megan

[4] Second correction! replaced X_i with ∞_i as in the last two parts

[5] thanks Jim van Meter for showing me this post on the n-Category Café, where I learned this definition

References

[Mun] James R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.