Math 6210 — Fall 2012
Assignment #2

Choose 10 of the problems below to submit by Weds., Sep. 19.[1] ←1

Exercise 1. Let X and Y be metric spaces and $f : X \to Y$ a function. We have two definitions of what it means for f to be continuous:

(a) f is continuous as a function between metric spaces if, for any $x \in X$ and any $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$;

(b) f is continuous if, when X and Y are given their metric topologies, $f^{-1}(U)$ is open for every open $U \subset Y$.

Show that these definitions are equivalent.

Exercise 2. [HY, Exercise 1-4].

(a) Show that the collection of all open half-spaces is a subbasis for the standard topology on $\mathbf{R}^n$. (A subset $S \subset \mathbf{R}^n$ is called an **open half-space** if there is an affine linear function $f : \mathbf{R}^n \to \mathbf{R}$ such that $S = \{x \in \mathbf{R}^n \mid f(x) > 0\}$. For f to be **affine linear** means that $f(x) = c + g(x)$ where $c \in \mathbf{R}$ is a constant and $g : \mathbf{R}^n \to \mathbf{R}$ is a linear function.)

(b) Show that you can still get a subbasis by taking the open half spaces defined over $\mathbf{Q}$. An open half space is defined over $\mathbf{Q}$ if it is given by an inequality $f(x) > 0$ with $f(x)$ an affine linear function defined over $\mathbf{Q}$; this means that $f(x) = c + g(x)$ where $c \in \mathbf{Q}$ and $g : \mathbf{R}^n \to \mathbf{R}$ is a linear function whose matrix representation has only rational entries.

Exercise 3. Let X be a set. Recall that a basis for a topology on X is a collection $\mathscr{C}$ of subsets of X such that a subset of X is open if and only if it is a union of elements of $\mathscr{C}$. This definition is not the same as [Mun, §13, Definition]. Show that this definition is equivalent to the one given in Munkres.

In other words, you should demonstrate that the statements below concerning a collection $\mathscr{C}$ of subsets of X are equivalent:

(a) There is a topology T on X such that a subset $U \subset X$ is open in T if and only if $U = \bigcup_i V_i$ for some collection of $V_i \in \mathscr{C}$.

(b) (i) For each $x \in X$ there is at least one $V \in \mathscr{C}$ with $x \in V$,[2] ←2

(ii) if $V, W \in \mathscr{C}$ and $x \in V \cap W$ then there is some $U \in \mathscr{C}$ with $x \in U \subset V \cap W$.

[1] this line was omitted in the original post

[2] typo corrected: some words were accidentally repeated

Exercise 4. Suppose that $f : X \to Y$ is a surjection of topological spaces and Y has the quotient topology. Must f be an open map? (Recall that for f to be **open** means that $f(U) \subset Y$ is open whenever $U \subset X$ is open.) Prove it or give a counterexample.

Exercise 5. Let $f : X \to Y$ be a continuous map of topological spaces. Check the following properties:

(a) Suppose that Y has the topology induced from X (the quotient topology). Let $g : Y \to Z$ be a function. Then gf is continuous if and only if g is.

(b) Suppose that X has the topology induced from Y (the generalized subspace topology). Let $h : W \to X$ be a function. Then fh is continuous if and only if h is.

Exercise 6. Consider the map $f : \mathbf{R} \to S^1$ defined by

$$f(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Let α be an irrational number and give $\alpha\mathbf{Z} \subset \mathbf{R}$ the subspace topology. Let $X \subset S^1$ be the image of the map

$$\alpha\mathbf{Z} \subset \mathbf{R} \to S^1.$$

Give X the subspace topology from S^1. Show that X **does not** have the quotient topology from the map

$$p : \alpha\mathbf{Z} \to X.$$

(Hint: show that p is injective.)

Exercise 7. Let $X = [0, 1]$, let $Y = [0, 1)$, and let $Z = (0, 1)$. Give all of these the subspace topology from $\mathbf{R}$. Show that no two of X, Y, and Z are homeomorphic.

Exercise 8. Let I be the interval $[0, 1]$. Let R be the equivalence relation on I wherein $x \equiv y$ if $x = y$ or $\left|x - y\right| = 1$.[3] That is $0 \equiv 1$ is the only equivalence ←3 other than $x \equiv x$ for all $x \in I$. Show that I/R, with its quotient topology, is homeomorphic to S^1. You may use the fact that $S^1 \cong \mathbf{R}/\mathbf{Z}$ without proof.[4] ←4

(Hint: You may find it helpful to prove that if U is an open subset of I that contains both 0 and 1 then $\bigcup_{n \in \mathbf{Z}}(n + U) \subset \mathbf{R}$ is open.)

Exercise 9. A **directed set** is a pair $(I, \leq)$ where I is a set and $\leq$ is a filtered pre-order on I. Recall that a **pre-order** is a relation $\leq$ on I that is *reflexive* ($\alpha \leq \alpha$ for all $\alpha \in I$) and *transitive* (if $\alpha \leq \beta \leq \gamma$ then $\alpha \leq \gamma$). Note that we can have $\alpha \leq \beta$ and $\beta \leq \alpha$ without $\alpha = \beta$, so a pre-order is not quite the same thing as a partial order.

The pre-order is said to be **filtered** if, for every pair $\alpha, \beta \in I$, there is some $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.

[3] typo corrected: originally I forgot to include the possibility $x = y$; thanks Jim van Meter

[4] added this suggestion

(a) [Mun, ch. 3, Supplementary Exercise #1(c)] Let X be a topological space containing a point x. Define I to be the set of open subsets of X that contain x. Give I the relation $U \leq V$ if $U \supset V$. Show that $(I, \leq)$ is a directed set.[5] ←5

Let X be a topological space. A **net** in X is a function $I \to X : \alpha \mapsto x_\alpha$ where I is a directed set. We say that $\{x_\alpha\}$ converges to $x \in X$ if, for any open set $U \subset X$ containing x, there is an index $\alpha \in I$ such that for all $\beta \geq \alpha$ we have $x_\beta \in U$.

(b) [Mun, ch. 3, Supplementary Exercise #6] Let X be a topoological space. Show that $Z \subset X$ is closed if and only if whenever $(x_\alpha)_{\alpha \in I}$ is a net in Z having a limit $x \in X$, the limit is actually contained in X.

(c) [Mun, ch. 3, Supplementary Exercise #7] Let X and Y be topological spaces and $f : X \to Y$ a function. Show that f is continuous if and only if f takes convergent nets to convergent nets (i.e., if $(x_\alpha)_{\alpha \in I}$ is convergent then $(f(x_\alpha))_{\alpha \in I}$ is also convergent).

(d) [Mun, ch. 3, Supplementary Exercise #10]. Let X be a topological space. Show that X is compact if and only if every net in X has a convergent subnet. (See the problem in [Mun] for a hint.)

Exercise 10. In the last problem set, we saw that there is a p-adic metric on $\mathbf{Q}$ defined by $d(x, y) = \left|x - y\right|_p$. We also saw that every metric space has a *completion*. The completion of $\mathbf{Q}$ in the p-adic metric is called $\mathbf{Q}_p$ and the completion of $\mathbf{Z}$ in the p-adic metric is called $\mathbf{Z}_p$ and is called the ring of **p-adic integers**.

(a) Suppose that for all $k \geq 0$ we have an integer a_k with $0 \leq a_k < p$. Show that the sequence of partial sums of the series[6] ←6

$$\sum_{k=0}^{\infty} a_k p^k = a_0 + a_1 p + a_2 p^2 + \cdots \tag{1}$$

form a Cauchy sequence in $\mathbf{Z}$ with respect to the p-adic metric. The sum therefore has a limit in $\mathbf{Z}_p$.

(b) Show that every element of $\mathbf{Z}_p$ can be represented as a sum of the form (1). Here are some hints:[7] ←7

(i) Let n be a positive integer. Show that the function $\mathbf{Z} \to \mathbf{Z}/p^n\mathbf{Z}$ is continuous if $\mathbf{Z}$ is given the p-adic topology and $\mathbf{Z}/p^n\mathbf{Z}$ is given the discrete topology.

[5] typo corrected: this said "net" before. Thanks Tom Reid.
[6] accidentally called this a sequence before; thanks to the person who noticed this
[7] added hints

(ii) Deduce that if x_i, $i = 1, 2, \ldots$ is a Cauchy sequence in $\mathbf{Z}$ (with respect to the p-adic metric) then the sequence $x_i \pmod{p^n\mathbf{Z}}$, $i = 1, 2, \ldots$ is covergent in $\mathbf{Z}/p^n\mathbf{Z}$. Let y_n be the limit of this sequence.

(iii) For each n, let $\widetilde{y}_n$ be the smallest positive representative for for y_n in $\mathbf{Z}$. Show that the sequence $\widetilde{y}_i$, $i = 1, 2, \ldots$ is Cauchy and is equivalent to $x_1, x_2, \ldots$.

(iv) Define $a_n = \frac{\widetilde{y}_{n+1} - \widetilde{y}_n}{p^n}$. Check that $\sum_{n=0}^{\infty} a_n p^n$ coverges to $\lim_{n\to\infty} x_n$ in $\mathbf{Z}_p$.

(c) Show that if x and y are two points in $\mathbf{Q}_p$ then there are disjoint open sets U and V in $\mathbf{Q}_p$ such that $x \in U$ and $y \in V$ and $U \cup V = \mathbf{Q}_p$. (Hint: show that $B(x, \epsilon)$ is open and closed for most values of ϵ.)

Conclude that $\mathbf{Q}_p$ is **totally disconnected**: for any $x \in \mathbf{Q}_p$, the only connected subset of $\mathbf{Q}_p$ containing x is $\{x\}$.[8] ←8

(d) Show that the topology on $\mathbf{Q}_p$ is *not discrete* by showing that the set $\{0\}$ is not open.

(e) Prove that $\mathbf{Z}_2$ is homeomorphic to the middle third Cantor set.

Exercise 11. Let X be the topological space

$$X = \Big(\{0\} \times [-1, 1]\Big) \cup \Big([-1, 1] \times \{0\}\Big) \subset \mathbf{R}^2,$$

with the subspace topology. Let Y be the topological space

$$Y = \Big(\{0\} \times [0, 1]\Big) \cup \Big([-1, 1] \times \{0\}\Big) \subset \mathbf{R}^2,$$

also with the subspace topology. Show that X and Y are not homeomorphic.

Exercise 12. Let $I = [0, 1]$ with the subspace topology from $\mathbf{R}$. Show that any continuous[9] function $f : I \to I$ must have a fixed point. (A **fixed point** is a point $x \in I$ such that $f(x) = x$.) ←9

(Hint: Consider $\{(x, f(x)) \big| x \in I\} \subset I \times I$. This is called the **graph** of f.)

References

[HY] John G. Hocking and Gail S. Young. *Topology.* Dover Publications Inc., New York, second edition, 1988.

[Mun] James R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[8] corrected the definition of totally disconnected

[9] typo corrected: I forgot the word "continuous"! Thanks Megan Ly and the person who asked about this in class (I apologize for forgetting who it was)