Math 6120 — Fall 2012
Assignment #1

Choose 10 of the problems below to submit by Weds., Sep. 5.

Exercise 1. [Mun, §21, #10]. Show that the following are closed subsets of $\mathbf{R}^2$:

(a) $A = \{(x,y) | xy = 1\}$,

(b) $S^1 = \{(x,y) | x^2 + y^2 = 1\}$, and

(c) $B^2 = \{(x,y) | x^2 + y^2 \leq 1\}$.

(Hint: use the fact that the pre-image of a closed set under a continuous map is closed. You don't have to prove that polynomial functions from $\mathbf{R}^n$ to $\mathbf{R}$ are continuous (but you should prove it for yourself if you haven't done a proof before).)

Exercise 2. Let X be a topological space and x a point in X. Define a new space X' whose underlying set is X and whose open subsets are the empty set and the open subsets of X containing x. Show that X' is also a topological space.

Exercise 3. Show that every metric on a finite set is always equivalent to the metric $d(x,y) = 1$. Conclude that a finite metric space has the discrete topology.

Exercise 4. (a) Suppose that X is a topological space with the **discrete** topology and Y is any other topological space. Show that any function $X \to Y$ is continuous.

(b) Suppose that X is a topological space and Y is a topological space with the **indiscrete** topology. Show that any function $X \to Y$ is continuous.

(c) Let P be a point (with the unique topology on a one-element set) and let X be a topological space. Conclude from the above that any functions $P \to X$ and $X \to P$ are continuous.

Exercise 5. Let (X, d) be a metric space.

(a) Fix a positive number t. Let $d'(x,y) = \min\{d(x,y), t\}$. Show that d' is also a metric on X.

(b) Let

$$d''(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Show that this is a metric on X.

(c) Show that both of the above metrics are equivalent to d.

Exercise 6. (a) Show that the following are metrics on $\mathbf{R}^n$:

(a) The Euclidean metric, $d(x,y) = \left|x - y\right|$. Cf. [Mun, §20, #9] for hints.

(b) The sup metric, $d'(x,y) = \sup_{i=1}^{n}\left\{\left|x_i - y_i\right|\right\}$.

(c) The metric $d''(x,y) = \sum_{i=1}^{n}\left|x_i - y_i\right|$.

(b) Show that all of the metrics described above are equivalent.

Exercise 7. Show that if d and d' are two *equivalent*[1] metrics on the same set X then ←1

(a) a sequence in X converges in d if and only if it converges in d', and

(b) the limits are the same.

Exercise 8. Let $\mathbf{Q}$ be the set of rational numbers and fix a prime number p. Every non-zero rational number can be written in a unique way as ap^n where n is an integer and a is a rational number that is prime to p.[2] Define

$$\left|ap^n\right|_p = p^{-n}$$
$$\left|0\right|_p = 0.$$

Show that

$$d(x,y) = \left|x - y\right|_p$$

defines a metric on $\mathbf{Q}$.

Exercise 9. [Mun, §20, #3]. Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbf{R}$ is continuous.

(b) Let X' be a topological space with the same underlying set as X. Show that if $d : X' \times X' \to \mathbf{R}$ is continuous then the topology of X' is finer than the topology of X.

Exercise 10. Suppose that X is a topological space, x is a point in X, and $x_1, x_2, \ldots$ is a sequence of points in X. Does either of the following statements imply the other? Are they equivalent?

(i) The point x is contained in the closure of the set $S = \left\{x_1, x_2, \ldots\right\}$ but not in S itself.

(ii) The point x is the limit of the sequence $x_1, x_2, \ldots$.

Give proofs or counterexamples to justify your answer.

[1] typo corrected—the word "equivalent" was missing; thanks Shawn

[2] This means that $a = \frac{x}{y}$ where x and y are integers that are not divisible by p.

Exercise 11. Let X be a topological space. Call a subset S of X closed in the sequential convergence (sc) topology if, whenever $x_1, x_2, \ldots$ is a sequence in S possessing a limit in X, the limit lies in S.

(a) Show that this is a topology on X.

(b) Let X' be the topological space whose underlying set is X, but given the sc topology. Let f be the map $X' \to X$ [3] whose underlying function is $\leftarrow_3$ the identity. Show that f is continuous.

(c) Demonstrate that the sc topology is not always the same as the original topology. You may want to use the following sequence of steps:

  (i) Let Y be an uncountable well-ordered set whose ordinality is equal to the first uncountable ordinal. Note that Y has no maximal element. Let $X = Y \cup \{\infty\}$ be the union of Y and a maximal element. Declare that $U \subset X$ is open if $U = \emptyset$ or if there is some $y \in Y$ such that $U = \{x \in X \,|\, x \geq y\}$.[4] Show that this is a topology on X. $\leftarrow_4$

  (ii) Let U be the complement of ∞ in X. Show that U is not closed.

  (iii) Show that the subset U of the last part is closed in the sc topology. (Hint: use the fact that for any increasing sequence of countable ordinals $x_1 < x_2 < x_3 < \cdots$ there is a countable ordinal y such that $x_i < y$ for all i; you can prove this fact by observing that a countable union of countable sets is countable.)

(d) Show that the sc topology is the same as the original topology if X satisfies the first countability axiom (each point of X has a countable basis of open neighborhoods).

Exercise 12. [Mun, §20, #6]. Let $\mathbf{x} = (x_1, x_2, \ldots)$ be an element of $\mathbf{R}^\omega$ and $\epsilon \in (0,1)$ a real number. Define

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots .$$

(a) Show that $U(\mathbf{x}, \epsilon) \neq B_d(\mathbf{x}, \epsilon)$ where d is the uniform metric on $\mathbf{R}^\omega$. (Hint: show that $U(\mathbf{x}, \epsilon)$ is not even open!)

(b) Prove that[5] $\leftarrow_5$

$$B_d(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

Exercise 13. Let X be a metric space. A *Cauchy sequence* in X is a sequence $x_1, x_2, \ldots$ such that for every $\epsilon > 0$ there is a positive integer N having the property that $d(x_n, x_m) < \epsilon$ for every $n, m \geq N$.

[3] typo corrected here

[4] this was not phrased correctly before; thanks James Van Meter for this observation and clarification suggestions

[5] typo corrected in the equation below; thanks Paul Lessard

(a) Let X' be the set of Cauchy sequences in X. Define a relation on X' by which $x_1, x_2, \ldots$ is related to $y_1, y_2, \ldots$ if, for any $\epsilon > 0$, there exists a positive integer N such that $d(x_n, y_n) < \epsilon$ for all $n \geq N$. Show that this is an equialence relation.

(b) Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two elements of X'. Define

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \lim_{n \to \infty} d(x_n, y_n).$$

(c) Show that this is a well-defined function on $X' \times X'$. (Show that the limit exists.)

(d) Show that this function is actually well-defined on the equivalence classes in $\overline{X}$ and makes $\overline{X}$ into a metric space.

(e) Show that two sequence $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are in the same equivalence class in X' if and only if $d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = 0$.

(f) Let $\overline{X}$ be the set of equivalence classes in X'. Let $i : X \to \overline{X}$ that sends $x \in X$ to the equivalence class of the sequence $x, x, x, \ldots$. Show that i is injective and that its image is dense in $\overline{X}$.

$\overline{X}$ is known as the **completion** of the metric space X.

Exercise 14. A pre-order on a set S is a relation $\rightsquigarrow$ that is reflexive and transitive. That is $x \rightsquigarrow x$ for all $x \in S$ and if $x \rightsquigarrow y \rightsquigarrow z$ then $x \rightsquigarrow z$.

(a) An element x of a topological space X is said to specialize to $y \in S$ if y is contained in the closure of the set $\{x\}$. We write $x \rightsquigarrow y$ to mean that y is a specialization of x. Show that a closed set contains the specializations of all its elements. A set that contains all the specializations of all its elements is said to be **closed under specialization**.

(b) Show that a subset of a **finite** topological space is closed **if and only if** it is closed under specialization.

(c) Give an example of an **infinite** topological space and a subset that is closed under specialization but is not closed.

(d) Show that there is a one-to-one correspondence between topologies on a finite set S and pre-orders on S. (Hint: Declare that $x \rightsquigarrow y$ if y is a specialization of x.)

(e) Compute the number of topologies on a set with 2 elements, up to re-ordering of the elements.

(f) Compute the number of topologies on a set with 3 elements, up to re-ordering.

References

[Mun] James R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.