Math 6120 — Fall 2012
Assignment #1

Choose 10 of the problems below to submit by Weds., Sep. 5.

Exercise 1. [Mun, §21, #10]. Show that the following are closed subsets of $\mathbf{R}^2$:

(a) $A = \{(x, y) | xy = 1\}$,

(b) $S^1 = \{(x, y) | x^2 + y^2 = 1\}$, and

(c) $B^2 = \{(x, y) | x^2 + y^2 \leq 1\}$.

(Hint: use the fact that the pre-image of a closed set under a continuous map is closed. You don't have to prove that polynomial functions from $\mathbf{R}^n$ to $\mathbf{R}$ are continuous (but you should prove it for yourself if you haven't done a proof before).)

Solution. The maps $f, g : \mathbf{R} \to \mathbf{R}^2$ defined by $f(x, y) = xy$ and $g(x, y) = x^2+y^2$ are continuous because they are polynomials. The sets $\{1\}$ and $(-\infty, 1]$ are closed intervals so they are closed in $\mathbf{R}$. Therefore $A = f^{-1}\{1\}$, $S^1 = g^{-1}\{1\}$, and $B^2 = g^{-1}(-\infty, 1]$ are all closed. □

Comments. Most people got this. One mistake that occurred was to try to use $f(A) = \{1\}$ rather than $f^{-1}\{1\} = A$, etc.; these are not equivalent, and continuity only guarantees that the pre-image of a closed set is closed, not that a set with closed image is closed. □

Exercise 2. Let X be a topological space and x a point in X. Define a new space X' whose underlying set is X and whose open subsets are the empty set and the open subsets of X containing x. Show that X' is also a topological space.

Solution. $\varnothing$ is open in X' by definition; X is open in X' because it is open in X and it contains x.

If U_i, $i \in I$ are open in X' then they are open in X so $\bigcup U_i$ is open in X. If the U_i are all empty then so is their union, so it is open in X'; if any U_i is non-empty then it contains x, so $x \in \bigcup U_i$ and $\bigcup U_i$ is open in X'.

If U and V are open in X' then they are open in X so $U \cap V$ is open in X. If either U or V is empty then so is their intersection, so it is open in X'. Otherwise both U and V are non-empty so x is contained in both, hence $x \in U \cap V$, so $U \cap V$ is open in X'. □

Comments. Most people got this problem. One common omission was to state explicitly that $\bigcup U_i$ and $U \cap V$ are open in X. Only two people dealt carefully with the empty set when taking unions and intersections. $\square$

Exercise 3. Show that every metric on a finite set is always equivalent to the metric[1] ←1

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Conclude that a finite metric space has the discrete topology.

Solution. Perhaps the most efficient proof is to show that all metrics on a finite set induce the discrete topology; since two metrics are equivalent if and only if they are induce the same topology, this proves in particular that every metric is equivalent to the given metric.

Let (X, d') be a finite metric space and set $\epsilon = \min\{d'(x,y) \mid x, y \in X \text{ and } x \neq y\}$. Because X is finite ϵ is a positive number. Then for every $x \in X$ the ball $B_{d'}(x, \epsilon)$ is open (because $\epsilon > 0$) and consists of the point x alone. It follows that d' induces the discrete topology. $\square$

Exercise 4. (a) Suppose that X is a topological space with the **discrete** topology and Y is any other topological space. Show that any function $X \to Y$ is continuous.

Solution. If $U \subset Y$ is open then $f^{-1}(U)$ is open because all subsets of X are open. $\square$

(b) Suppose that X is a topological space and Y is a topological space with the **indiscrete** topology. Show that any function $X \to Y$ is continuous.

(c) Let P be a point (with the unique topology on a one-element set) and let X be a topological space. Conclude from the above that any functions $P \to X$ and $X \to P$ are continuous.

Comments. Almost everyone who attempted it got this problem. One mistake was to attempt to use preservation of limits as the definition of continuity; while continuous functions always preserve convergent sequences, the converse is not always true. $\square$

Exercise 5. Let (X, d) be a metric space.

(a) Fix a positive number t. Let $d'(x,y) = \min\{d(x,y), t\}$. Show that d' is also a metric on X.

[1] I forgot to include $d(x,x) = 0$ originally; thanks James Van Meter

Solution. We have $d'(y,x) = \min\{d(y,x), t\} = \min\{d(x,y), t\} = d(x,y)$; the middle equality holds because d is a metric.

If $d'(x,y) = 0$ then $\min\{d(x,y), t\} = 0$. Since $t > 0$ we must therefore have $d(x,y) = 0$.

For $x, y, z \in X$ we have

$$\begin{gathered} d'(x,z) \le d(x,z) \le d(x,y) + d(y,z) \\ d'(x,z) \le t \le \min\{d(x,y) + t, t + d(y,z), 2t\}. \end{gathered}$$

Therefore

$$\begin{aligned} d'(x,z) &\le \min\{d(x,y) + d(y,z), d(x,y) + t, t + d(y,z), 2t\} \\ &\le \min\{d(x,y), t\} + \min\{d(y,z), t\} \\ &= d'(x,y) + d'(y,z). \end{aligned}$$

□

(b) Let

$$d''(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Show that this is a metric on X.

Solution. We have $d''(y,x) = d''(x,y)$ for obvious reasons. If $d''(x,y) = 0$ then $d(x,y) = 0$ so $x = y$. For the triangle inequality, note first that if a and b are non-negative real numbers then $a \le b$ if and only if $\frac{a}{1+a} \le \frac{b}{1+b}$. We therefore have

$$\begin{aligned} d'(x,z) &= \frac{d(x,z)}{1 + d(x,z)} \\ &\le \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)} \\ &= \frac{d(x,y)}{1 + d(x,y) + d(y,z)} + \frac{d(y,z)}{1 + d(x,y) + d(y,z)} \\ &\le \frac{d(x,y)}{1 + d(x,y)} + \frac{d(y,z)}{1 + d(y,z)} \\ &= d'(x,y) + d'(y,z) \end{aligned}$$

□

(c) Show that both of the above metrics are equivalent to d.

Solution. The balls of radii $< t$ form a basis in any metric topology, provided $t > 0$. These are the same in the d and d' metrics.

Since $a \leq b$ if and only if $\frac{a}{1+a} \leq \frac{b}{1+b}$, we have

$$B_d(x, \epsilon) = B_{d''}(x, \frac{\epsilon}{1+\epsilon}).$$

Therefore the d and d'' metrics have the same open balls, hence give the same topology. □

Exercise 6. (a) Show that the following are metrics on $\mathbf{R}^n$:

(i) The Euclidean metric, $d(x, y) = \left|x - y\right|$. Cf. [Mun, §20, #9] for hints.

Solution. We check the triangle inequality only since the other properties are trivial. It's enough to show that

$$d(x, z)^2 \leq d(x, y)^2 + 2d(x, y)d(y, z) + d(y, z)^2.$$

Let $a = x - y$, $b = y - z$, and $c = x - z$. Then we are trying to show

$$\left|a + b\right|^2 \leq (\left|a\right| + \left|b\right|)^2.$$

Expanding we see that this is equivalent to the Cauchy–Schwarz inequality:

$$a \cdot b \leq \left|a\right|\left|b\right|.$$

One way to prove this is to remark that $a \cdot b = \left|a\right|\left|b\right| \cos(\theta)$ where θ is the angle between the vectors a and b and $\left|\cos(\theta)\right| \leq 1$. □

(ii) The sup metric, $d'(x, y) = \sup_{i=1}^{n}\{\left|x_i - y_i\right|\}$.

Solution. We have $d'(x, y) = \sup\{\left|x_i - y_i\right|\} = \sup\{\left|y_i - x_i\right|\} = d'(y, x)$ and if $d'(x, y) = 0$ then $\left|x_i - y_i\right| = 0$ for all i, i.e., $x = y$. We also have

$$\begin{aligned} d'(x, z) = \sup\{\left|x_i - z_i\right|\} &\leq \sup\{\left|x_i - y_i\right| + \left|y_i - z_i\right|\} \\ &\leq \sup\{\left|x_i - y_i\right|\} + \sup\{\left|y_i - z_i\right|\} = d'(x, y) + d'(y, z). \end{aligned}$$

□

(iii) The metric $d''(x, y) = \sum_{i=1}^{n}\left|x_i - y_i\right|$.

Solution. If d'' is symmetric because $\left|x_i - y_i\right|$ is symmetric in x and y; if $d''(x, y) = 0$ then all of the positive numbers $\left|x_i - y_i\right|$ must be zero so $x_i = y_i$ for all i. Triangle inequality:

$$\begin{aligned} d''(x, z) = \sum_{i=1}^{n}\left|x_i - z_i\right| &\leq \sum_{i=1}^{n}\left|x_i - y_i\right| + \left|y_i - z_i\right| \\ &= d''(x, y) + d''(y, z). \end{aligned}$$

□

(b) Show that all of the metrics described above are equivalent.

Solution. We have $d'(x, y) \leq d(x, y) \leq d''(x, y) \leq nd'(x, y)$. ☐

Exercise 7. Show that if d and d' are two *equivalent*[2] metrics on the same set X then ←2

(a) a sequence in X converges in d if and only if it converges in d', and

(b) the limits are the same.

Solution. A continuous function takes convergent sequences to convergent sequences and preserves limits. The identity function gives a continuous function $(X, d) \to (X, d')$ that is its own continuous inverse $(X, d') \to (X, d)$. ☐

Solution. Here's another solution: convergence of a sequence to a point is a topological property. Equivalent metrics give rise to the same topology. ☐

Comments. Many of you made this proof more complicated than necessary by trying to apply the definition of convergence directly. ☐

Exercise 8. Let $\mathbf{Q}$ be the set of rational numbers and fix a prime number p. Every non-zero rational number can be written in a unique way as ap^n where n is an integer and a is a rational number that is prime to p.[3] Define

$$\begin{aligned} \left|ap^n\right|_p &= p^{-n} \\ \left|0\right|_p &= 0. \end{aligned}$$

Show that

$$d(x, y) = \left|x - y\right|_p$$

defines a metric on $\mathbf{Q}$.

Solution. First, the only rational number z such that $\left|z\right|_p = 0$ is $z = 0$ so if $\left|x - y\right|_p = 0$ then $x = y$. If $x - y = ap^n$ then $y - x = -ap^n$ and $\left|x - y\right|_p = \left|y - x\right|_p$.

Now we check the triangle inequality: Suppose that $x - y = ap^n$ and $y - z = bp^m$ so that $d(x, y) = p^{-n}$ and $d(y, z) = p^{-m}$. Then

$$d(x, z) = \left|ap^n + bp^m\right|_p.$$

The value depends on whether n or m is larger and by symmetry we may as well assume that $n \leq m$. Then

$$d(x, z) = \left|p^n(a + bp^{m-n})\right|_p.$$

[2]typo corrected—the word "equivalent" was missing; thanks Shawn

[3]This means that $a = \frac{x}{y}$ where x and y are integers that are not divisible by p.

I claim that $a+bp^{m-n}$ is of the form cp^k for some $k \geq 0$ and c prime to p. This will imply that $d(x,z) = \left|cp^{n+k}\right|_p = p^{-n-k}$ which is less than $d(x,y)+d(y,z) = p^{-n}+p^{-m}$.

Let's check the claim: Write $a = \frac{a_1}{a_2}$ and $b = \frac{b_1}{b_2}$. Then

$$\frac{a_1}{a_2} + \frac{b_1}{b_2}p^{m-n} = \frac{a_1 b_2 + a_2 b_1 p^{m-n}}{a_2 b_2}.$$

The denominator is a product of numbers prime to p so is prime to p. The numerator is an integer so it is divisible by a non-negative power of p. This proves the claim. $\square$

Comments. A *very* common mistake was to attempt to prove that $\left|x+y\right|_p = \max\{\left|x\right|_p, \left|y\right|_p\}$ by writing $x = ap^n$ and $y = bp^n$, assuming by symmetry that $n \leq m$, and writing

$$\left|x+y\right|_p = \left|(a+bp^{m-n})p^n\right|_p = \left|a+bp^{m-n}\right|_p p^{-n}.$$

Then saying that $a+bp^{m-n}$ is prime to p. Of course, it is prime to p if $m > n$, but if $m = n$ it very well may be divisible by p.

Note also that $\left|x+y\right|_p = \max\{\left|x\right|_p, \left|y\right|_p\}$ is inequivalent with the fact that $\left|x+(-x)\right|_p = 0$. $\square$

Exercise 9. [Mun, §20, #3]. Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbf{R}$ is continuous.

Solution. Suppose that $(x,y) \in X \times X$. Supposing that $\epsilon > 0$ we would like to show that there is an open neighborhood U of (x,y) such that $\left|d(x',y') - d(x,y)\right| < \epsilon$ for $(x',y') \in U$. For U we take the open set $U = B(x,\epsilon/2) \times B(y,\epsilon/2)$ consisting of all (x',y') such that $d(x,x') < \epsilon/2$ and $d(y,y') < \epsilon/2$. For all $(x',y') \in U$ we have

$$d(x',y') \leq d(x,y) + d(x,x') + d(y,y') \leq d(x,y) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = d(x,y) + \epsilon$$

and similarly, with the roles of (x',y') and (x,y) reversed, we have $d(x,y) \leq d(x',y') + \epsilon$. Therefore $\left|d(x,y) - d(x',y')\right| < \epsilon$. $\square$

(b) Let X' be a topological space with the same underlying set as X. Show that if $d : X' \times X' \to \mathbf{R}$ is continuous then the topology of X' is finer than the topology of X.

Solution. We must show that for any $x \in X$ and any $\epsilon > 0$ the ball $B_d(x,\epsilon)$ is open in X'. Consider the composition of maps

$$X' \xrightarrow{f} X' \times X' \xrightarrow{d} \mathbf{R}$$

where $f(y) = (x, y)$. Let $g : X' \to X'$ be the composition $d \circ f$. Then f is continuous and d is continuous by assumption so g is continuous. Therefore $g^{-1}((-\infty, \epsilon))$ is open in X'. But $g^{-1}((-\infty, \epsilon))$ consists of all $y \in X'$ such that $d(f(y)) < \epsilon$ and $d(f(y)) = d(x, y)$. That is $g^{-1}((-\infty, \epsilon)) = B_d(x, \epsilon)$ is open in X'. □

Exercise 10. Suppose that X is a topological space, x is a point in X, and $x_1, x_2, \ldots$ is a sequence of points in X. Does either of the following statements imply the other? Are they equivalent?

(i) The point x is contained in the closure of the set $S = \{x_1, x_2, \ldots\}$ but not in S itself.

(ii) The point x is the limit of the sequence $x_1, x_2, \ldots$.

Give proofs or counterexamples to justify your answer.

Solution. Neither implies the other. The sequence $(n, \sin(1/n))$ in $\mathbf{R}^2$ does not converge to any limit, but the closure of $\{(n, \sin(1/n)\}$ contains the interval $\{0\} \times [-1, 1]$. Also, the constant sequence $x_n = x$ has limit x but x is contained in $\{x_n | n \in \mathbf{Z}\}$. □

Exercise 11. Let X be a topological space. Call a subset S of X closed in the sequential convergence (sc) topology if, whenever $x_1, x_2, \ldots$ is a sequence in S possessing a limit in X, the limit lies in S.

(a) Show that this is a topology on X.

Solution. Suppose $\{S_i\}$ is a collection of sc-closed subsets of X. We want to show that $\bigcap S_i$ is also sc-closed. Let $x_1, x_2, \ldots$ be a sequence of elements of $\bigcap S_i$ and suppose that $x \in X$ is a limit of the x_i. Then x lies in each S_i and therefore in their intersection. Therefore sc-closed subsets are closed under arbitrary intersection.

Now suppose that S and T are sc-closed subsets of X. Suppose that $x_1, x_2, \ldots$ is a sequence of elements of $\bigcup S_i$ and suppose that $x \in X$ is a limit of the x_i. We wish to show that $x \in S \cup T$. Each x_i must come either from S or T. Therefore one of S or T must contain infinitely many of the x_i. Let us suppose it is S, the argument being essentially the same if it is T, and let $y_1, y_2, \ldots$ be the subsequence of those x_i that lie in S. Recall that for the x_i to converge to x means that every open $U \subset X$ containing x also contains all but finitely many of the x_i. Therefore it must also contain all but finitely many of the y_i as well. Thus the y_i form a subsequence of the x_i that also converges to x. Since S is closed, this means that $x \in S \subset S \cup T$. □

(b) Let X' be the topological space whose underlying set is X, but given the sc topology. Let f be the map $X' \to X$ [4] whose underlying function is the identity. Show that f is continuous. ←4

[4]typo corrected here

Solution. Let S be closed in X (in the usual topology). Then S contains the limit of every sequence drawn from S so S is also sc-closed. Every closed set is also sc closed, so the sc-topology is *finer* than the given topology and the map $X' \to X$ is therefore continuous. $\square$

(c) Demonstrate that the sc topology is not always the same as the original topology. You may want to use the following sequence of steps:

(i) Let Y be an uncountable well-ordered set whose ordinality is equal to the first uncountable ordinal. Note that Y has no maximal element. Let $X = Y \cup \{\infty\}$ be the union of Y and a maximal element. Declare that $U \subset X$ is open if $U = \emptyset$ or if there is some $y \in Y$ such that $U = \{x \in X \,|\, x \geq y\}$.[5] Show that this is a topology on X. $\leftarrow$[5]

(ii) Let U be the complement of ∞ in X. Show that U is not closed.

(iii) Show that the subset U of the last part is closed in the sc topology. (Hint: use the fact that for any increasing sequence of countable ordinals $x_1 < x_2 < x_3 < \cdots$ there is a countable ordinal y such that $x_i < y$ for all i; you can prove this fact by observing that a countable union of countable sets is countable.)

Solution. We would like to find a topological space X with a subset S that contains the limit of every sequence in S but is still not closed in X. Let $X = \aleph_1 \cup \{\infty\}$ with the convention that $\infty > x$ for all $x \in \aleph_1$. For each $x \in X$, let U_x be the set of all $z \in X$ with $z > x$. Declare that the U_x are open for all $x \in X$.

It is obvious that an arbitrary union of open subsets is open. Also, $U_x \cap U_y = U_{\max\{x,y\}}$ so finite unions of open subsets are also open. Therefore this is a topology on X.

Let U be the complement of ∞ in X. Then $\overline{U} = X$. Indeed, $Z \subset X$ is closed if and only if there exists an $x \in X$ such that Z consists of all $z \leq x$ in X. There is no such element in U so U is not closed. As it is missing only one element of X, its closure must be all of X.

Now, consider a sequence $x_1, x_2, \ldots$ in U. I claim that this sequence cannot converge to ∞. Indeed, each x_i is a countable ordinal. There is therefore a countable ordinal that is larger than all of them (a countable union of countable sets being countable). Call this element y. Then U_y does not contain any of the x_i but still is an open neighborhood of ∞ in X. Therefore U is sc-closed but not closed. $\square$

(d) Show that the sc topology is the same as the original topology if X satisfies the first countability axiom (each point of X has a countable basis of open neighborhoods).

[5]this was not phrased correctly before; thanks James Van Meter for this observation and clarification suggestions

Exercise 12. [Mun, §20, #6]. Let $\mathbf{x} = (x_1, x_2, \ldots)$ be an element of $\mathbf{R}^\omega$ and $\epsilon \in (0,1)$ a real number. Define

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots .$$

(a) Show that $U(\mathbf{x}, \epsilon) \neq B_d(\mathbf{x}, \epsilon)$ where d is the uniform metric on $\mathbf{R}^\omega$. (Hint: show that $U(\mathbf{x}, \epsilon)$ is not even open!)

Solution. Consider the point $\mathbf{y}$ with coordinates $y_n = x_n + \frac{n-1}{n}\epsilon$. Then $\mathbf{y} \in U(\mathbf{x}, \epsilon)$ but we show that $U(\mathbf{x}, \epsilon)$ contains no open neighborhood of $\mathbf{y}$. Indeed, for any $\delta > 0$, choose a positive k such that $\delta > \frac{\epsilon}{k}$. If $n \geq k$ then $y_n + \frac{\epsilon}{k} > \epsilon$ so the point $\mathbf{z}$ with coordinates $z_n = y_n + \frac{\epsilon}{k}$ is in $U(\mathbf{y}, \delta)$ but is not in $U(\mathbf{x}, \epsilon)$. □

(b) Prove that[6] ←6

$$B_d(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

Solution. If $\mathbf{y} \in U(\mathbf{x}, \delta)$ for some $\delta < \epsilon$ then $\sup\{|y_n - x_n|\} \leq \delta$. Therefore $U(\mathbf{x}, \delta) \subset B_d(\mathbf{x}, \epsilon)$ for all $\delta < \epsilon$, which proves one inclusion.

Conversely, if $\mathbf{y} \in B_d(\mathbf{x}, \epsilon)$ then $\sup\{|x_n - y_n|\} < \epsilon$ so there is some $\delta < \epsilon$ such that $|x_n - y_n| < \delta$ for all n. Therefore $\mathbf{y} \in U(\mathbf{x}, \delta)$. This proves the reverse inclusion. □

Exercise 13. Let X be a metric space. A *Cauchy sequence* in X is a sequence $x_1, x_2, \ldots$ such that for every $\epsilon > 0$ there is a positive integer N having the property that $d(x_n, x_m) < \epsilon$ for every $n, m \geq N$.

(a) Let X' be the set of Cauchy sequences in X. Define a relation on X' by which $x_1, x_2, \ldots$ is related to $y_1, y_2, \ldots$ if, for any $\epsilon > 0$, there exists a positive integer N such that $d(x_n, y_n) < \epsilon$ for all $n \geq N$. Show that this is an equialence relation.

Solution. Reflexivity and symmetricity are evident so they are omitted. If $x \sim y \sim z$ then there exist positive integers N and M such that $d(x_n, y_n) < \epsilon/2$ for $n \geq N$ and $d(y_n, z_n) < \epsilon/2$ for $n \geq M$. Then for $n \geq \max\{N, M\}$,

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) = \epsilon.$$

Thus $\sim$ is transitive. □

(b) Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two elements of X'. Define

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \lim_{n \to \infty} d(x_n, y_n).$$

[6] typo corrected in the equation below; thanks Paul Lessard

(c) Show that this is a well-defined function on $X' \times X'$. (Show that the limit exists.)

Solution. We show that the sequence is Cauchy: choose N such that $d(x_n, x_m) < \epsilon/2$ for $n, m \geq N$, and $d(y_n, y_m) < \epsilon/2$ for $n, m \geq N$. Then for $n, m \geq N$ we have

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < d(x_m, y_m) + \epsilon \\ d(x_m, y_m) &\leq d(x_n, x_m) + d(x_n, y_n) + d(y_n, y_m) < d(x_n, y_n) + \epsilon. \end{aligned}$$

Therefore

$$\left| d(x_n, y_n) - d(x_m, y_m) \right| < \epsilon$$

and the sequence $(d(x_n, y_n))_{n \in \mathbf{N}}$ is Cauchy, hence has a limit. □

(d) Show that this function is actually well-defined on the equivalence classes in $\overline{X}$ and makes $\overline{X}$ into a metric space.

Solution. From the definition, $x \sim y$ if and only if $\lim d(x_n, y_n) = 0$. It is immediate that $d(x, y) = d(y, x)$. We also have

$$d(x, z) = \lim d(x_n, z_n) \leq \lim(d(x_n, y_n) + d(y_n, z_n)) = d(x, y) + d(y, z).$$

Thus if $x \sim y$ we have $d(x, z) \leq d(x, y) + d(y, z) = d(y, z)$ and similarly $d(y, z) \leq d(x, z)$. Therefore $d(x, z) = d(y, z)$ and d is well-defined on equivalence classes. Along the way we have also demonstrated that d is a metric on $\overline{X}$. □

(e) Show that two sequence $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are in the same equivalence class in X' if and only if $d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = 0$.

Solution. Shown above. □

(f) Let $\overline{X}$ be the set of equivalence classes in X'. Let $i : X \to \overline{X}$ that sends $x \in X$ to the equivalence class of the sequence $x, x, x, \ldots$. Show that i is injective and that its image is dense in $\overline{X}$.

Solution. Suppose $d(i(x), i(y)) = 0$. Then $\lim d(i(x)_n, i(y)_n) = 0$. But $d(i(x)_n, i(y)_n)$ is the constant sequence with value $d(x, y)$ so $d(x, y) = 0$, hence $x = y$ because d is a metric on X. (It is also immediate that d is continuous because $d(i(x), i(y)) = d(x, y)$.)

If x is a Cauchy sequence in X representing an element of $\overline{X}$ then the sequence of points $i(x_n)$ have limit x in $\overline{X}$. Indeed,

$$\lim_{n \to \infty} d(i(x_n), x) = \lim_n \lim_m d(i(x_n), x_m) = \lim_{n,m \to \infty} d(x_n, x_m)$$

which is zero because the sequence is Cauchy. □

$\overline{X}$ is known as the **completion** of the metric space X.

Exercise 14. A pre-order on a set S is a relation $\rightsquigarrow$ that is reflexive and transitive. That is $x \rightsquigarrow x$ for all $x \in S$ and if $x \rightsquigarrow y \rightsquigarrow z$ then $x \rightsquigarrow z$.

(a) An element x of a topological space X is said to specialize to $y \in S$ if y is contained in the closure of the set $\{x\}$. We write $x \rightsquigarrow y$ to mean that y is a specialization of x. Show that a closed set contains the specializations of all its elements. A set that contains all the specializations of all its elements is said to be **closed under specialization**.

Solution. Suppose S is closed, $x \in S$, and $x \rightsquigarrow y$. Then $y \in \overline{\{x\}}$, which is contained in S because S is closed. Therefore $y \in S$. $\square$

(b) Show that a subset of a **finite** topological space is closed **if and only if** it is closed under specialization.

Solution. Suppose $S \subset X$ and X is finite. We already know that S is closed under specialization if S is closed so assume that S is closed under specialization. If $x \in \overline{S}$ then every open neighborhood of x meets S. Let U be the intersection of all open neighborhoods of x. This is a finite collection of subsets (since the collection of all open subsets of X is finite) so it is open. Therefore x has a smallest open neighborhood. Let $y \neq x$ be an element of this smallest open neighborhood of x that is also contained in S. Then $y \rightsquigarrow x$ for otherwise there would be an open subset of x that does not contain y. Therefore $x \in S$. We conclude that $S = \overline{S}$ and S is closed. $\square$

(c) Give an example of an **infinite** topological space and a subset that is closed under specialization but is not closed.

Solution. Let X be any topological space in which points are closed. Then any subset of X is closed under specialization, but we can certainly find an example of such a space with a non-discrete topology (for example, $X = \mathbf{R}$). $\square$

(d) Show that there is a one-to-one correspondence between topologies on a finite set S and pre-orders on S. (Hint: Declare that $x \rightsquigarrow y$ if y is a specialization of x.)

Solution. Given a topology on S, define the pre-order as in the hint. We check that it is a pre-order. Clearly x is in its own closure, so we only have to check that $x \rightsquigarrow y \rightsquigarrow z$ implies that $x \rightsquigarrow z$. That is, we have to check that if $y \in \overline{\{x\}}$ and $z \in \overline{\{y\}}$ then $z \in \overline{\{x\}}$. But if $y \in \overline{\{x\}}$ then $\overline{\{y\}} \subset \overline{\{x\}}$ so if z is in $\overline{\{y\}}$ then it is in $\overline{\{x\}}$, as we wish.

Now suppose that we are given a pre-order on S. We define a topology. Declare that a subset $Z \subset S$ is closed if whenever $x \in Z$, every specialization of x is also in Z. To check that this is a topology on S we have to check the two defining properties: if $\{Z_i\}$ are closed under specialization then so are their intersection and union. Indeed, if $x \in \bigcap Z_i$ then $x \in Z_i$ for all i; if in addition $x \rightsquigarrow y$ then $y \in Z_i$ for all i (since each Z_i is closed under specialization), hence $y \in \bigcap Z_i$. Similarly, if $x \in \bigcup Z_i$ and $x \rightsquigarrow y$ then $x \in Z_i$ for some i, so $y \in Z_i$ (since Z_i is closed under specialization), so $y \in \bigcup Z_i$ as well.

We have defined functions

$$F : \{\text{topologies on } S\} \to \{\text{pre-orders on } S\}$$
$$G : \{\text{pre-orders on } S\} \to \{\text{topologies on } S\}$$

that we must show are mutually inverses. Supposing that T is a topology on S we demonstrate that $G(F(T)) = T$ by showing that a subset Z of S is closed in the topology T if and only if it is closed in the topology $G(F(T))$. Since $Z \subset S$ is closed in $G(F(T))$ if and only if it is closed under specialization, we are supposed to check that $Z \subset S$ is closed (in T) if and only if it is closed under specialization. We checked this earlier.

Therefore $G \circ F$ is the identity. We can finish the proof by showing that $F \circ G$ is also the identity. That is, given a pre-order $\rightsquigarrow$ on S, we have to show that $F(G(\rightsquigarrow)) = \rightsquigarrow$. By definition, $F(G(\rightsquigarrow))$ is the pre-order $\succ$ wherein $x \succ y$ if y lies in the closure of x in the topology $G(\rightsquigarrow)$. But by definition, the closure of $\{x\}$ in $G(\rightsquigarrow)$ is the smallest subset of S that contains x and is closed under specialization. I claim that this is precisely the set Z of all specializations of x: indeed, if $y \in Z$ then y is a specialization of x, so if z is a specialization of y, then z is also a specialization of x (transitivity of specialization), hence $z \in Z$. We conclude from this $x \succ y$ if and only if y is a specialization of x. In other words, $\succ$ and $\rightsquigarrow$ are the same pre-order. □

(e) Compute the number of topologies on a set with 2 elements, up to reordering of the elements.

Solution. This is the same as the number of ways of pre-ordering a set with 2 elements:

(a) no relations,
(b) $1 \rightsquigarrow 2$,
(c) $1 \rightsquigarrow 2$ and $2 \rightsquigarrow 1$.

□

(f) Compute the number of topologies on a set with 3 elements, up to reordering.

Solution. Again, we count pre-orders:

(a) no relations (indicrete topology),

(b) $1 \rightsquigarrow 2$,

(c) $1 \rightsquigarrow 2 \rightsquigarrow 3$,

(d) $1 \rightsquigarrow 2$, $1 \rightsquigarrow 3$,

(e) $1 \rightsquigarrow 3$, $2 \rightsquigarrow 3$,

(f) $1 \rightsquigarrow 2$, $2 \rightsquigarrow 1$,

(g) $1 \rightsquigarrow 2 \rightsquigarrow 3$, $2 \rightsquigarrow 1$, $2 \rightsquigarrow 3$,

(h) $1 \rightsquigarrow 2 \rightsquigarrow 3$, $3 \rightsquigarrow 2$,

(i) $1 \rightsquigarrow 2 \rightsquigarrow 3 \rightsquigarrow 2 \rightsquigarrow 1$ (discrete topology).

$\square$

References

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