Math 6210 — Fall 2012
Exam #2

Due Wednesday, December 18. Cite any sources you use (including people with whom you discuss the exam.

Problem 1. Give an example of a continuous *surjective*[1] map of topological spaces $f : B' \to B$, with B' ←[1]
connected,[2] and a *connected* covering space $p : E \to B$ such that the covering space $p' : E' = E \times_B B' \to B'$ ←[2]
is *not connected*.

Problem 2. Compute the homology of an oriented surface of genus g (a g-holed torus).

Problem 3. Prove that $\mathbf{R}^n$ is homeomorphic to $\mathbf{R}^m$ if and only if $n = m$.

Problem 4. (a) Prove that for $n \geq 1$ there is no continuous retraction from the closed ball D^n onto its boundary S^{n-1}.

(b) Prove that every continuous map $f : D^n \to D^n$ has a fixed point.

Problem 5. Fill in the details of the following proof of the fundamental theorem of algebra.

A continuous map $f : X \to Y$ with the property that $f^{-1}(A)$ is compact for every compact subset $A \subset Y$ is called **proper**.

(a) Show that if $f : X \to Y$ is proper and Y is Hausdorff[3] and has a neighborhood basis of compact subsets ←[3]
then f is closed.

(b) Let $f : \mathbf{C} \to \mathbf{C}$ be a polynomial map. Show that f is proper.

(c) Let $f : \mathbf{C} \to \mathbf{C}$ be a *non-constant* polynomial map. Show that f is open.

You may prove this however you like, but a simple proof is possible using the Cauchy integral formula. Another suggested route for a proof:

(i) Let a be an element of $\mathbf{C}$. Choose a small loop γ around a in $\mathbf{C}$. Show that $f(\gamma)$ is a loop around
$f(a)$. (Hint: write $f(z) = f(a) + c(z-a)^n(1 + (z-a)g(z))$[4] for some $c \in \mathbf{C}$ and polynomial g.) ←[4]
Let U be the interior of γ and V the interior of $f(\gamma)$.

(ii) Suppose that b is an element of V such that there is no $a' \in U$ with $f(a') = b$. Obtain a contradiction by considering the map $U \to V \smallsetminus \{b\}$.

(d) Conclude that a non-constant polynomial map is surjective, and therefore that there is some $z \in \mathbf{C}$ such that $f(z) = 0$.

Problem 6. [Hat, §2.2, #32] Let X be a topological space and SX its suspension (see [Hat, p. 8]). Prove that $H_n(SX, \mathbf{Z}) \cong H_{n-1}(X, \mathbf{Z})$ for all $n \geq 2$. Determine formulas for $H_1(SX, \mathbf{Z})$ ahd $H_0(SX, \mathbf{Z})$ in terms of $H_1(X, \mathbf{Z})$ and $H_0(X, \mathbf{Z})$.

References

[Hat] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[1] added this word!
[2] added this requirement!
[3] the Hausdorff hypothesis was omitted originally
[4] corrected the suggestion by adding $f(a)$