Math 6210 — Fall 2012
Exam #1

Due Friday, October 26. Remember to cite any sources you use.

Problem 1. A topological space X is called path connected if for every pair of points x and y there is a path $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

(a) Show that if X is path connected then X is connected.

(b) Let X be the union of $\{(x,y) \in \mathbf{R}^2 \mid x \neq 0, y = \sin(1/x)\}$[1] and $\{(0,0)\}$. Show that X is connected but not path connected. $\leftarrow_1$

(c) Show that the point $(0,0)$ in the example X above does not have a neighborhood basis of compact subsets of X.

Problem 2. Let X be a topological space. Construct a new space $X' = X \cup \{\infty\}$ in which a subset $U \subset X'$ is called open if it is an open subset of X or if it contains ∞ and its complement is a closed,[2] compact subset of X.[3] $\leftarrow_2$ $\leftarrow_3$

(a) Show that this is a topology on X' and that the inclusion map $X \to X'$ is continuous.

(b) Show that X' is compact.

(c) Let Y be the subspace of X' whose underlying set is X. Show that the map $X \to Y$ is a homeomorphism. (Show, in other words, that the subspace topology on X induced from X' is the same as the original topology on X.)

Problem 3. Let $\mathbf{R}P^n$ be the quotient of S^n by the action of $\{\pm 1\}$. Give $\mathbf{R}P^n$ the quotient topology.

(a) Show that the map $p : S^n \to \mathbf{R}P^n$ is a covering space.

(b) Prove that $\pi_1(\mathbf{R}P^n, *) \cong \mathbf{Z}/2\mathbf{Z}$ for $n \geq 2$ and $\pi_1(\mathbf{R}P^1, *) \cong \mathbf{Z}$.[4] $\leftarrow_4$

You may use the following fact without proof: If $n \geq 2$ and $f : [0,1] \to S^n$ is a continuous map then there exists a homotopy h, relative to the endpoints 0 and 1, between f and a continuous map $g : [0,1] \to S^n$ where g is not surjective.[5] $\leftarrow_5$

Problem 4. Let X be a topological space and $PX = \mathrm{Cont}([0,1], X)$ its **path space**. Let $\varphi : X \to PX$ be the function that sends $x \in X$ to the constant path at x. Show that φ is a homotopy equivalence.

Problem 5. Let C be the Cantor set and let I be the unit interval.

(a) Show that by definition C is the subset of all real numbers in $[0,1]$ whose ternary (base 3) expansion does not contain a 1 after the decimal point.

(b) Define a function $f : C \to [0,1]$ as follows: Suppose that $x \in C$ has ternary expansion $0.a_1a_2a_3\cdots$. Then let $f(x)$ be the number in $[0,1]$ with *binary* expansion $0.b_1b_2b_3\cdots$ where $b_i = 0$ if $a_i = 0$ and $b_i = 1$ if $a_i = 2$. Show that this function is well-defined, continuous, and surjective. (When showing this is well-defined, remember that the ternary expansion of a number is not unique!)

(c) Prove that $C \times C$ and C are homeomorphic. (Hint: use something like $g(0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots) = 0.a_1b_1a_2b_2a_3b_3\cdots$, but be careful to make sure it is well defined.)

[1] correction: added the hypothesis $x \neq 0$, which was omitted before; thanks Megan
[2] correction: "closed" forgotten originally; thanks Dmitro
[3] This problem had quite a few issues. I couldn't find a reasonable way to salvage part (d), so I deleted it.
[4] correction: separated the case $n = 1$; thanks Megan
[5] clarification about what is supposed not to be surjective; thanks Jim

(d) Let $g : C \to C \times C$ be a homeomorphism. Show that $h = (f \times f) \circ g$ is surjective.

(e) Extend h to a continuous function $I \to I^2$. (Hint: use linear interpolation.)

(f) Prove that there is a continuous surjection from I to I^n for all $n \geq 0$.

(g) Conclude that there is also a continuous surjection from S^1 to S^n for all n.