Math 6210 — Fall 2012
Exam #1

Due Friday, October 26. Remember to cite any sources you use.

Problem 1. A topological space X is called path connected if for every pair of points x and y there is a path $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

(a) Show that if X is path connected then X is connected.

Solution. Suppose $X = Y \amalg Z$. Suppose that Y is non-empty and pick $y \in Y$. For any $x \in X$ there is a path $f : [0,1] \to X$ connecting x to y. Since f is continuous and $[0,1]$ is connected, the image of f is a connected subset of X, therefore is contained in Y. But then x is contained in Y. This holds for all $x \in X$ so $X = Y$ and $Z = \varnothing$. This applies to any decomposition of X and $Y \amalg Z$, so X must be connected. $\square$

(b) Let X be the union of $\{(x,y) \in \mathbf{R}^2 \mid x \neq 0, y = \sin(1/x)\}$[1] and $\{(0,0)\}$. Show that X is connected but not path connected. ←1

Solution. Let $U = X \cap (-\infty, 0) \times \mathbf{R}$. The sequence $(-\frac{1}{k\pi}, 0)$ in U converges to $(0,0)$ so $\overline{U} = U \cup \{(0,0)\} = X \cap (-\infty, 0] \times \mathbf{R}$.

Since U is the continuous image of the function $f(t) = (t, \sin(1/t))$ it is connected. Therefore its closure is connected [**?**, 7 (c)]. The same argument shows that $\overline{V} = X \cap [0, \infty)$ is connected. But then $\overline{U} \cap \overline{V} = \{(0,0)\}$ is non-empty so $X = \overline{U} \cup \overline{V}$ is connected by [**?**, 7 (a)].

Now we show X is not path connected. Suppose there were a path f in X connecting $(1/\pi, 0)$ to $(0,0)$. Since $[0,1]$ is compact, the image of f is compact, hence closed. Composing with first projection $\mathbf{R}^2 \to \mathbf{R}$, we see that any path from $(1, \sin(1))$ to $(0,0)$ must induce a path from 1 to 0 in $\mathbf{R}$. Since the interval is connected, such a path must be surjective. Therefore, for every $x \in (0,1]$ there must be some $t \in [0,1]$ such that $f(t) = (x, \sin(1/x))$. In particular, the image of f must contain the points $x = (\frac{1}{\pi/2 + 2k\pi}, 1)$ for all integers $k \geq 0$. But the image of f is supposed to be closed, so it would also have to contain $(0,1)$, which is the limit of that sequence of points. Since $(0,1)$ is not in X, this is a contradiction. $\square$

(c) Show that the point $(0,0)$ in the example X above does not have a neighborhood basis of compact subsets of X.

Solution. Let $U = (-1/2, 1/2) \times (-1/2, 1/2) \cap X$ and suppose $(0,0) \in V \subset K \subset U$ with V open and K compact. Then K is a compact subset of $\mathbf{R}^2$, hence is closed in $\mathbf{R}^2$. Choose a real number $y \in (0, \epsilon)$. Then for each sufficiently large integer k we have $1/2\pi k \in (0, \epsilon)$ and $1/(\pi/2 + 2\pi k) \in (0, \epsilon)$. By the intermediate value theorem, there is therefore some $x_k \in (1/(\pi/2 + 2\pi k), 1/2\pi k)$ with $\sin(1/x_k) = y$. We therefore have a sequence $(x_k, y) \in K$ with limit $(0, y)$, which does not converge to an element of X. Therefore K is not closed in $\mathbf{R}^2$, hence is not compact. $\square$

Problem 2. Let X be a topological space. Construct a new space $X' = X \cup \{\infty\}$ in which a subset $U \subset X'$ is called open if it is an open subset of X or if it contains ∞ and its complement is a closed,[2] compact subset of X.[3] ←2 ←3

(a) Show that this is a topology on X' and that the inclusion map $X \to X'$ is continuous.

[1] correction: added the hypothesis $x \neq 0$, which was omitted before; thanks Megan
[2] correction: "closed" forgotten originally; thanks Dmitro
[3] This problem had quite a few issues. I couldn't find a reasonable way to salvage part (d), so I deleted it.

Solution. Since $\varnothing$ is an open subset of X it is an open subset of X'. Since the complement of X' is the empty set, which is a compact subset of X, the set X' is open as well.

Suppose that U_i are open subsets of X'. Let $U = \bigcup U_i$. If all U_i are contained in X then their union is also contained in X and is open because a union of open sets is open. If on the other hand $\infty \in U$ then $\infty \in U_j$ for some j. Therefore $X \smallsetminus U_j$ is compact. Then $X \smallsetminus U = \bigcap (X \smallsetminus U_i)$ is a closed subset of X that is contained in $X \smallsetminus U_j$. But $X \smallsetminus U_j$ is compact, so $X \smallsetminus U$ is compact (and closed, as already remarked). Therefore U is an open subset of X by definition.

Finally, suppose that U and V are open subsets of X'. Then $X \smallsetminus (U \cap V) = (X \smallsetminus U) \cup (X \smallsetminus V)$ is a finite union of closed, compact subsets of X, hence is closed and compact.

Now we check that $i : X \to X'$ is continuous. Suppose that $U \subset X'$ is open. Then either $U \subset X$, in which case $i^{-1}(U) = U$, and U is an open subset of X. If U contains ∞ then $X \smallsetminus U \subset X$ is closed in X, so $i^{-1}(U) = U \cap X$ is open in X. Thus $i^{-1}(U)$ is open in X for all open $U \subset X'$. This shows that i is continuous. □

(b) Show that X' is compact.

Solution. Suppose $X' = \bigcup U_i$. We must have $\infty \in U_j$ for some j. Let $Z = X \smallsetminus U_j$. By definition of the open subsets of X', the subset Z is compact. Then the U_i cover Z, and since Z is compact, finitely many of the U_i—say $U_{i_1}, \ldots, U_{i_k}$ suffice. Then U_j and $U_{i_1}, \ldots, U_{i_k}$ together constitute a finite subcover of $\{U_i\}$. □

(c) Let Y be the subspace of X' whose underlying set is X. Show that the map $X \to Y$ is a homeomorphism. (Show, in other words, that the subspace topology on X induced from X' is the same as the original topology on X.)

Solution. We already know that the map $X \to Y$ is a continuous bijection. On the other hand, it is open because if $U \subset X$ is open then $U \subset X'$ is open. An open, continuous bijection is a homeomorphism. □

Problem 3. Let $\mathbf{R}\mathrm{P}^n$ be the quotient of S^n by the action of $\{\pm 1\}$. Give $\mathbf{R}\mathrm{P}^n$ the quotient topology.

(a) Show that the map $p : S^n \to \mathbf{R}\mathrm{P}^n$ is a covering space.

Solution. Choose a cover of S^n by open subsets U with the property that $-U \cap U = \varnothing$. Then for each such U we have $p^{-1}(p(U)) = -U \cup U$, which is open in S^n, so $p(U)$ is open in $\mathbf{R}\mathrm{P}^n$. Therefore the sets $p(U)$ form an open cover $\mathbf{R}\mathrm{P}^n$. Moreover, for each such U, the map $U \to p(U)$ is a bijection: it is surjective by definition and injective because $U \cap -U = \varnothing$. Therefore $p^{-1}(p(U)) \cong \{\pm 1\} \times p(U)$. This shows that p is a covering space. □

(b) Prove that $\pi_1(\mathbf{R}\mathrm{P}^n, *) \cong \mathbf{Z}/2\mathbf{Z}$ for $n \geq 2$ and $\pi_1(\mathbf{R}\mathrm{P}^1, *) \cong \mathbf{Z}$.[4] ←[4]

Solution. If $n \geq 2$ then S^n is a simply connected covering space of $\mathbf{R}\mathrm{P}^n$ so $\pi_1(\mathbf{R}\mathrm{P}^n)$ acts simply transitively on $p^{-1}(*)$, which is a 2-element set. Therefore $\pi_1(\mathbf{R}\mathrm{P}^n)$ is a group with 2 elements, and there is only one such up to isomorphism.

If $n = 1$ we construct a homeomorphism $S^1 \cong \mathbf{R}\mathrm{P}^1$. Consider the map $q : S^1 \to S^1$ given by $x \mapsto x^2$ when S^1 is viewed as the set of complex numbers of absolute value 1. Note that we have a commutative diagram

$$\begin{tikzcd}
\mathbf{R} \arrow[r, "f"] \arrow[dr, "g"'] & S^1 \arrow[d, "q"] \\
& S^1
\end{tikzcd}$$

[4]correction: separated the case $n = 1$; thanks Megan

where the two maps $f, g : \mathbf{R} \to S^1$ are both open quotient maps. This means that the map $S^1 \to S^1$ is a quotient map ($q^{-1}(U)$ is open iff $f^{-1}q^{-1}(U)$ is open iff U is open). It also implies that q is open, since $q(U) = q(f(f^{-1}(U)))$ and qf is open. Now, note that if $p(x) = p(y)$ then $q(x) = q(y)$ (i.e., $x = \pm y$ if and only if $x^2 = y^2$). Therefore there is a well-defined function $r : S^1 \to \mathbf{R}\mathrm{P}^1$ such that $rq = p$. By the universal property of the quotient topology, r is continuous. It is also open: $r(U) = r(q(q^{-1}(U))) = p(q^{-1}(U))$ is the image of the open set $q^{-1}(U)$ under the open map p.

This is injective: say $r(x) = r(y)$; then pick x', y' such that $x = q(x')$ and $y = q(y')$; then $p(x') = rq(x') = rq(y') = p(y')$ so $x' = \pm y'$ so ${x'}^2 = {y'}^2$ so $x = q(x') = q(y') = y$. It is also surjective: if $x \in \mathbf{R}\mathrm{P}^1$ then $x = p(x')$ for some $x' \in S^1$ so $x = rq(x')$ so x is in the image of r.

Thus r is an open, continuous bijection, hence a homeomorphism. It follows that $\pi_1(\mathbf{R}\mathrm{P}^1) \cong \pi_1(S^1) = \mathbf{Z}$. □

You may use the following fact without proof: If $n \geq 2$ and $f : [0,1] \to S^n$ is a continuous map then there exists a homotopy h, relative to the endpoints 0 and 1, between f and a continuous map $g : [0,1] \to S^n$ where g is not surjective.[5] ←5

Problem 4. Let X be a topological space and $PX = \mathrm{Cont}([0,1], X)$ its **path space**. Let $\varphi : X \to PX$ be the function that sends $x \in X$ to the constant path at x. Show that φ is a homotopy equivalence.

Solution. We should first verify that φ is continuous. This is because the projection map $X \times [0,1] \to X : (x,t) \mapsto x$ is continuous. Therefore by the universal property of the path space, the map $\varphi : X \to PX$ sending x to the function $\varphi(x)$ with $\varphi(x)(t) = x$ for all $t \in [0,1]$ is a continuous function.

Let $\psi : PX \to X$ be the function sending a path $f : [0,1] \to X$ to $f(0)$. We verify ψ is continuous: by the universal property of PX (which applies because $[0,1]$ is compact and Hausdorff), there is a continuous map $PX \times [0,1] \to X$ sending (f,t) to $f(t)$ (corresponding to the identity map $PX \to PX$). We also have a continuous map $PX \to PX \times [0,1]$ sending f to $(f, 0)$ by the universal property of the product. The map ψ is the composition of these two maps.

Now, we have $\psi \circ \varphi(x) = x$ so what we have to check is that $\varphi \circ \psi$ is homotopic to the identity. Let $F : PX \times [0,1] \to X$ be the universal map described above. Consider the map $\alpha : [0,1] \times [0,1] \to [0,1]$ defined by $\alpha(s,t) = st$. Then $H = F \circ (\mathrm{id} \times \alpha)$ defines a continuous map $H : PX \times [0,1] \times [0,1] \to X$ with $H(f,s,t) = f(st)$. By the universal property of the path space, we obtain a continuous map $h : PX \times [0,1] \to PX$ with $h(f,s)(t) = H(f,s,t) = f(st)$. But now, $h(f,0)(t) = f(0) = \varphi \circ \psi(f)(t)$ and $h(f,1)(t) = f(t) = \mathrm{id}(f)(t)$. Therefore h is a homotopy between $\varphi \circ \psi$ and the identity. □

Problem 5. Let C be the Cantor set and let I be the unit interval.

(a) Show that by definition C is the subset of all real numbers in $[0,1]$ whose ternary (base 3) expansion does not contain a 1 after the decimal point.

Solution. Let $\omega = \mathbf{Z}_{>0}$ be the set of positive integers. Give the set $\{0,2\}$ the discrete topology. Let $X = \{0,2\}^\omega$ with the product topology. Consider the map $\varphi : X \to [0,1]$ defined by

$$\varphi(a_1, a_2, \ldots) = \sum_{k=1}^{\infty} \frac{a_k}{3^k}.$$

Lemma. The map φ defined above is continuous and defines a homeomorphism from $\{0,2\}^\omega$ onto C.

Proof. Let $a = (a_k)_{k \in \omega}$ be a point of X. To prove continuity, we must show that, given any $\epsilon > 0$, there is an open neighborhood U of $(a_k)_{k \in \omega}$ such that if $b = (b_k)_\omega \in U$ then $\left|\varphi(a) - \varphi(b)\right| < \epsilon$. Now, choose an N such that $3^{-N} < \epsilon$. Let U be the set of $b = (b_k)_{k \in \omega}$ such that $b_k = a_k$ for all $k \leq N$. Note that U is open by definition of the product topology. Then

$$\left|\varphi(a) - \varphi(b)\right| = \Big|\sum_{k=N+1}^{\infty} \frac{a_k - b_k}{3^k}\Big| \leq \sum_{k=N+1}^{\infty} \frac{2}{3^k} = 3^{-N}.$$

[5] clarification about what is supposed not to be surjective; thanks Jim

Now we show that the image of f is contained in C. The Cantor set has the following recursive structure: $C = \bigcap_{n=0}^{\infty} C_n$ where $C_0 = [0,1]$ and $C_{n+1} = \frac{1}{3}C_n \cup (\frac{2}{3} + \frac{1}{3}C_n)$. We will show by induction that for each $a \in X$ we have $\varphi(a) \in C_n$ for all n. This is obvious for $n = 0$. Assume then that $\varphi(a) \in C_n$ for all $a \in X$. Let b be the element of X obtained by shifting: $b_k = a_{k+1}$ for all $k \in \omega$. Then $\varphi(b) \in C_n$ by induction, and $\varphi(a)$ is either $\frac{1}{3}\varphi(b)$ or it is $\frac{2}{3} + \frac{1}{3}\varphi(b)$. Either way, $\varphi(a)$ is in either $\frac{1}{3}C_n$ or $\frac{2}{3} + \frac{1}{3}C_n$ so it is in C_{n+1}.

Since C has the subspace topology from $[0,1]$, the universal property of the subspace topology implies that φ defines a continuous map $X \to C$. To show it is bijective, we now construct an inverse $g : C \to X$. We define g by induction. Let $x_0 = x$. Inductively define $g(x)_n = 0$ if $x_n \in [0, 1/3]$ and $g(x)_n = 2$ if $x_n \in [2/3, 1]$; set $x_n = 3x_n - g(x)_n$. Note that by the inductive structure of the Cantor set, $x_n \in C$ for all n, so this definition is legitimate. It is easy to verify that φ and g are inverse to one another, so I will omit this.

Finally, we must check that g is continuous (or equivalently, that φ is open). Let a be a point of X. There is a basis of open neighborhoods of a of the form $U_N = \{b \in X \mid a_k = b_k \text{ for } k \le N\}$. But $g^{-1}(U_N) = \varphi(U_N)$ is the interval

$$\left[\sum_{k=1}^{N} \frac{a_k}{3^k}, \frac{1}{3^N} + \sum_{k=1}^{N} \frac{a_k}{3^k}\right] \cap C.$$

This is open in C, since it coincides with

$$\left(-\frac{1}{3^N} + \sum_{k=1}^{N} \frac{a_k}{3^k}, \frac{2}{3^N} + \sum k = 1^N \frac{a_k}{3^k}\right) \cap C.$$

□

It now follows immediately that every element of the Cantor set can be expressed uniquely as $\sum_{k=1}^{\infty} \frac{a_k}{3^k}$ with $a_k \in \{0, 2\}$ for all k. □

(b) Define a function $f : C \to [0,1]$ as follows: Suppose that $x \in C$ has ternary expansion $0.a_1a_2a_3\cdots$. Then let $f(x)$ be the number in $[0,1]$ with *binary* expansion $0.b_1b_2b_3\cdots$ where $b_i = 0$ if $a_i = 0$ and $b_i = 1$ if $a_i = 2$. Show that this function is well-defined, continuous, and surjective. (When showing this is well-defined, remember that the ternary expansion of a number is not unique!)

Solution. Since the ternary expansion of an element of the Cantor set is unique the function indicated above is unique. We compose f with the homeomorphism $\varphi : X \to C$ defined in the lemma to get a map $f \circ \varphi : X \to [0,1]$ which is continuous if and only if f is. But

$$f \circ \varphi(a) = \sum_{k=1}^{\infty} \frac{a_k/2}{3^k}.$$

The proof of the continuity of f goes through exactly as in the proof of the lemma. □

(c) Prove that $C \times C$ and C are homeomorphic. (Hint: use something like $g(0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots) = 0.a_1b_1a_2b_2a_3b_3\cdots$, but be careful to make sure it is well defined.)

Solution. Note that $C \times C \cong X \times X = \{0,2\}^\omega \times \{0,2\}^\omega$. We therefore wish to show that $\{0,2\}^\omega \times \{0,2\}^\omega \cong \{0,2\}^\omega$.

First observe that for any topological space Y and any sets A and B, we have $Y^A \times Y^B \cong Y^{A \amalg B}$. Therefore

$$\{0,2\}^\omega \times \{0,2\}^\omega \cong \{0,2\}^{\omega \amalg \omega}.$$

But now we observe that as sets ω and $\omega \amalg \omega$ are in bijection: they are both countably infinite. □

Solution. Here is a second solution that is more direct.

Let g be as defined above. It is well-defined since the ternary expansion of an element of C is unique. It is obviously bijective. We only need to check that it is open and continuous. For this it will be preferable to work with a convenient subbasis for the topology of C.

Lemma. Sets of the form $[a, a + 3^{-N}] \cap C$, where $a = \sum_{k=1}^{N} \frac{a_k}{3^k}$ and $a_k \in \{0, 2\}$ for all k, form a basis for the topology of C.

Proof. First note that the sets $[a, a+3^{-N}]$ are all open because if a is of the form above then there is an $\epsilon > 0$ such that $(a-\epsilon, a+3^{-N}+\epsilon) \cap C = [a, a+3^{-N}]$. If $x = \sum_{k=1}^{\infty} \frac{x_k}{3^k}$ is in C then we can take $x_k \in \{0, 2\}$ for all k. Let $\epsilon > 0$ be any positive number. Note that for each N, we have $x \in [x_N, x_N + 3^{-N}]$ where $x_N = \sum_{k=1}^{N} \frac{x_k}{3^k}$. If we choose N large enough then $3^{-N} < \frac{\epsilon}{2}$ so $[x_N, x_N + 3^{-N}] \subset B(x, \epsilon)$. □

Note that the set $[a, a + 3^{-N}]$ of the lemma consists of all elements $x = \sum \frac{x_k}{3^k}$ of C such that $x_k = a_k$ for $k \leq N$. To show that g is continuous, we have to show that $g^{-1}([a, a + 3^{-N}])$ is open in $C \times C$. But $g^{-1}([a, a + 3^{-N}])$ consists of all $(\sum \frac{x_k}{3^k}, \frac{y_k}{3^k})$ such that $x_k = a_{2k-1}$ for $2k - 1 \leq N$ and $y_k = a_{2k}$ for $2k \leq N$. Thus $g^{-1}([a, a + 3^{-N}]) = [b, b + 3^{-M}] \times [c, c + 3^{-L}]$ where M is the largest integer such that $2M - 1 \leq N$ and L is the largest integer such that $2L \leq N$. By the lemma and the definition of the product topology, this is an open subset of $C \times C$. Therefore g is continuous.

Conversely, note that the set $[b, b + 3^{-M}] \times [c, c + 3^{-M}]$ with $b = \sum_{k=1}^{M} \frac{b_k}{3^k}$ and $c = \sum_{k=1}^{M} \frac{c_k}{3^k}$ form a basis for the topology of $C \times C$. And we have

$$g([b, b + 3^{-M}] \times [c, c + 3^{-M}]) = [g(b, c), g(b, c) + 3^{-2M}]$$

so g is also open. □

(d) Let $g : C \to C \times C$ be a homeomorphism. Show that $h = (f \times f) \circ g$ is surjective.

Solution. Since h is the composition of a surjection $f \times f$ and a bijection g it must be surjective. □

(e) Extend h to a continuous function $I \to I^2$. (Hint: use linear interpolation.)

Solution. Since C is closed in I, the Tietze extension theorem applies to give an extension of $h : C \to I^2$ to $\widetilde{h} : I \to I^2$. However, we didn't prove the Tietze extension theorem in class, so I'll give some direct arguments below. □

Solution. Since C is compact, for each $x \in I$ that is not in C there is a greatest $y \in C$ such that $y \leq x$ and a least $z \in C$ such that $x \leq z$. Define

$$\widetilde{h}(x) = \frac{x - y}{z - y} h(z) + \frac{z - x}{z - y} h(y).$$

This agrees with the given function $\widetilde{h}$ on C. We only have to show that h is continuous.

We show that h is continuous at each point of I. If $x \in I \smallsetminus C$ then $\widetilde{h}$ is linear near x so $\widetilde{h}$ is obviously continuous there. We therefore only need to check $\widetilde{h}$ is continuous on C.

Suppose $x \in C$ and $\epsilon > 0$. Since h is continuous, we can choose $\delta > 0$ such that if $y \in C$ and $\left|x - y\right| < \delta$ then $\left|h(x) - h(y)\right| < \frac{\epsilon}{2}$. If we choose δ so that $x - \delta$ and $x + \delta$ do not lie in C then $C \cap (x - \delta, x + \delta) = C \cap [x - \delta, x + \delta]$ is a closed subset of I, hence is compact. Therefore $C \cap (x - \delta, x + \delta)$ has a least element, which we call a, and a greatest element, b. Let a' be the greatest element of C that is less than a and let b' be the smallest element of C greater than b (which exist for the same reason we could find a and b). Since $\widetilde{h}$ is continuous on $[a', a]$ we can find $\alpha > 0$ such that if $y \in (a - \alpha, a]$

then $\left|\widetilde{h}(y) - h(a)\right| < \frac{\epsilon}{2}$. Likewise, we can find β such that if $y \in [b, b+\beta)$ then $\left|\widetilde{h}(y) - h(b)\right| < \frac{\epsilon}{2}$. Thus if $y \in (a-\alpha, b+\beta)$ then either $y \in (a-\alpha, x]$ or $y \in [x, b+\beta)$. In the former case,

$$\begin{aligned} \left|\widetilde{h}(y) - \widetilde{h}(x)\right| &\le \left|\widetilde{h}(y) - \widetilde{h}(a)\right| + \left|h(a) - h(x)\right| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

The case $y \in [x, b+\beta)$ is essentially identical. Therefore $\widetilde{h}$ is continuous at x. As this holds for all $x \in C$, and we have already shown $\widetilde{h}$ continuous for $x \in I \setminus C$, it follows that $\widetilde{h}$ is continuous. $\square$

(f) Prove that there is a continuous surjection from I to I^n for all $n \ge 0$.

Solution. Let h be a continuous surjection from I onto I^2. Then the composition

$$I \xrightarrow{h} I^2 \xrightarrow{h\times\mathrm{id}} I^3 \xrightarrow{h\times id} I^4 \xrightarrow{h\times\mathrm{id}} \cdots \xrightarrow{h\times\mathrm{id}} I^n$$

is a composition of continuous surjections, hence is a continuous surjection from I onto I^n. $\square$

(g) Conclude that there is also a continuous surjection from S^1 to S^n for all n.

Solution. Note that $h(0)$ and $h(1)$ both lie on the boundary of I^2. Therefore if $h_n : I \to I^n$ is the map constructed in the last part then $h_n(0)$ and $h_n(1)$ both lie on ∂I^n. Therefore if $p : I^n \to I^n/\partial I^n = S^n$ is the projection then $p \circ h_n$ factors set-theoretically through $I/\partial I = S^1$. But by definition of the quotient topology, this means that the induced function $S^1 \to S^n$ is continuous. Moreover, because $p \circ h_n$ is surjective (the composition of surjective functions) so must be the induced map $S^1 \to S^n$. $\square$

Solution. We have a continuous surjection of I^n onto S^n by collapsing the boundary. We can also find a continuous surjection $S^1 \to I$, e.g., the function $f(x, y) = 2x - 1$, where we view S^1 as a subset of $\mathbf{R}^2$ with coordinates x, y. The composition

$$S^1 \to I \to I^n \to S^n$$

is therefore a composition of continuous surjections, hence is a continuous surjection. $\square$