Contents

Part I

Algebraic manifolds

0 Before the beginning

0.1 Recommended background

Reading 0.1. [AM69, Chapters 1–3, 7], [Vak14, Sections 1.1–1.4]

Familiarity with commutative algebra at the level of [AM69, Chapters 1–3, 7], as well as basic point set topology, are essential. While category theory is not strictly necessary, some familiarity is strongly recommended, at the level of [Vak14, Sections 1.1–1.3]. At a minimum, you should be comfortable with universal properties. I will try to recall basic facts about limits and colimits as we need them, but if you haven't already encountered limits and colimits, you may want to study [Vak14, Section 1.4] too.

Acquanintance with differential geometry or complex analysis (especially Riemann surfaces) may be helpful, but is not essential.

0.2 References and how to use these notes

Each section of these notes is meant to correspond to one lecture, but these notes are not meant to be a complete reference for the course. Their main purpose is to help me organize the topics we will cover and to summarize what I want to say in lecture. You will need to consult other sources. In most cases, I have given a list of other references at the beginning of each section and in the table of contents.

Sometimes the references will cover more material than we do in lecture. It's always a good idea to look at this other material, but you may encounter some concepts we haven't defined in class. As a rule of thumb, you can skip parts of the reading that aren't mentioned in these notes.

I will draw a lot of the course material from Vakil's *Foundations of Algebraic Geometry* [Vak14]. This book is excellent, and if we had more I might have attempted to follow it linearly. As it is, we are going to jump around quite a lot, which is why I am using these notes to try to keep things organized.

In many places, the presentation in the notes won't be quite the same as the presentation in Vakil's book. One of the major differences is that I am going to spend more time on the functor of points. I'm going to trust you to keep the different approaches straight, but please let me know if things get muddled.

You might want to consult some other texts in case you find their presentation more compelling. Here are some suggestions:

(i) Hartshorne's *Algebraic Geometry* [Har77] is the classic reference. It is a bit terse, and a majority of the content is in the exercises.

(ii) Mumford's *Red Book of Varieties and Schemes* [Mum99] is a very good place to look for intuition. It is less complete than other references.

(iii) The *Stacks Project* [Sta15] is a definitive reference for an increasingly complete list of topics in algebraic geometry. Completeness and generality are often prioritized over readability, so the Stacks Project works well as a reference for specific results but less well as a textbook.

0.3 Goals for this course

The main goal for this course is to give you, the student, enough background to read a paper or advanced text in algebraic geometry, or to follow an algebraic geometry seminar. Secondarily, I hope to introduce you to enough algebraic geometry to participate in a summer research project, if you are interested.

The course is structured around a few theorems that I hope will provide motivation for the subject, which may otherwise be kind of technical.

0.4 Exercises

You need to do exercises to learn algebraic geometry. You need to do a whole lot of exercises, many more than I could possibly grade. I will grade some, though, and I will use these to determine your final grade. Here is what I expect:

(i) You should submit at least 3 exercises per week via D2L (learn.colorado.edu). Except for pictures, they have to be written in TEX.

(ii) The exercises you do are generally up to you (but see below), but must be relevant to the topics we are studying. Suggested sources are [Vak14], [Har77], and these notes, but you can get them from anywhere as long as they are relevant. You can even make up your own exercises if you want.

(iii) Occasionally I will give longer, guided assignments that will be required. These will be assigned about once every two weeks. They appear in the sections of the text labelled by a letter.

(iv) I will grade based on correctness, but also on your selection of appropriate exercises that are improving your understanding.

(v) Feel free to discuss assignments with anyone you like, and to consult any references you like. However, if want to use a theorem from a source other than [Vak14], [Har77], or these notes, please give the statement of the theorem in addition to the citation so that I don't have to dig up the reference.

(vi) You *must* cite any resources you use. You don't have to cite every source you glance at in the process of writing your solutions, but if consulting a reference contributes to your solution to the problem, you need to cite it. Otherwise you are effectively taking credit for someone else's work, and that is the definition of plagiarism. If you fail to cite something you should and I notice it, I will be very unhappy and deal with it harshly.

    It's good practice to make your citations as precise as possible, so get in the habit of always referencing numbered statements or page numbers in your citations.

0.5 Acknowledgements

Thanks to all of the students who discovered and corrected errors in this text. Specific acknowledgements appear with each correction. Thanks also to Shawn Burkett for fixing a frustrating error in my LATEXcode.

Chapter 1

Introduction to algebraic geometry

1 Bézout's theorem

Question 1.1. How many points do two algebraic plane curves have in common?

The answer to this question is Bézout's theorem. We will discuss several formulations of this theorem and a sketch of the proof. Our first goal in the course will be to make these statements, and the proof outlined below, precise.

By an (affine) algebraic plane curve, we will mean the set of solutions to a polynomial $f(x, y)$ in two variables. We can assume the coefficients of f are real numbers and that we are looking for solutions in $\mathbf{R}^2$, although in a moment we will want to look for solutions in $\mathbf{C}^2$ (and at that point we might as well allow coefficients in $\mathbf{C}$ as well).

The first example of an algebraic plane curve is a line. A line is given by a polynomial $ax + by + c$ where a and b are not both zero. In other words a line is given by a polynomial f of degree 1. (Degree of a polynomial in x and y is measured by giving both x and y degree 1.)

Exercise 1.2. Show that any line can be parameterized algebraically as $(x(t), y(t))$.

If f and g define plane curves C and D, then $C \cap D$ is the set of points (x, y) such that $f(x, y) = g(x, y) = 0$.

We can try some examples. If C and D are both lines then $C \cap D$ almost always consists of exactly one point. If the lines are parallel then we get usually get no solutions, but if the lines are the same, we get infinitely many solutions.

If $\deg C = 1$ and $\deg D = d$ then parameterize C by $(x(t), y(t))$. The intersection points correspond to the values of t such that $g(x(t), y(t)) = 0$. This is a degree d polynomial in t, so we expect d solutions—at least if we look in $\mathbf{C}$.

However, we don't always get d solutions, even when $d = 2$:

(i) Suppose $g(x, y) = x^2 + y^2 - 1$ and $y(t) = 1$ and $x(t) = t$. Then $g(x(t), y(t)) = t^2$ has just one solution (in any field).

(ii) Suppose $g(x, y) = xy$ and $x(t) = t$ and $y(t) = 0$. Then there are infinitely many solutions (any value of t).

(iii) Suppose that $g(x, y) = xy - 1$ and $x(t) = t$ and $y(t) = 1$. Then $g(x(t), y(t)) = 1$ there are no solutions at all.

What is going on geometrically? In the first case, we have a tangency. But suppose we move the line a little. Take $x(t) = t$ and $y(t) = s$. For different values of s we get different lines, and as long as s is near to but not equal to 1, we get two points of intersection. Thus we expect that *most* curves C and D (whatever that means), won't have a tangency, and this phenomenon won't occur.

In the second example, the line is a *component* of the curve D and we get infinitely many intersections. Again, we can try moving the line. If we try something like $x(t) = t$ and $y(t) = s(t + 1)$ then for $s \neq 0$ but near 0, we have two intersection points. Once again, we will be able to say that most curves C and D, don't share a component, so this phenomenon also does not usually occur. (Technically, this example is a special case of the previous one: It is a tangency of infinite order.)

In the last example, there is only one point of intersection when we expect two. Geometrically, we can see that C is parallel to an asymptote of D. If we deform C a little, say by taking $x(t) = t$ and $y(t) = 1 + st$ then as long as s is close but not equal to zero we get two solutions. As $s \to 0$, one of the solutions escapes to infinity. Once again, we can say that most lines intersect D in two points.

In view of these observations, *we exclude pairs of curves C and D that are tangent, share components, or share asymptotes* in Question 1.1.

More subtly, we have seen that making small changes to our curves C and D does not change the number of points of intersection between them, as long as the small changes do not introduce tangencies, common components, or common asymptotes. That is, *if C_t and D_t are one-parameter families of curves such that for no value of t do C_t and D_t have a tangency, common component, or common asymptote, then $|C_t \cap D_t|$ is constant.*

The final ingredient in a proof of Bézout's theorem will be to observe that *for any curves C and D (satisfying our assumptions) there is a 1-parameter family C_t and D_t (also satisfying our assumptions for each value of t) with $C_0 = C$ and $D_0 = D$ such that C_1 consists of $\deg C$ lines and D_1 consists of $\deg D_1$ lines.*

We can compute very easily that $|C_1 \cap D_1| = \deg(C)\deg(D)$. Putting all of these observations together, we have

$$|C \cap D| = |C_1 \cap D_1| = \deg(C)\deg(D).$$

This proves Bézout's theorem:

Theorem 1.3 (Bézout's theorem in the affine plane)**.** *For most algebraic plane curves C and D we have $|C \cap D| = \deg(C)\deg(D)$.*

'Most curves' may be interpreted to include curves that are not tangent, do not have parallel asymptotes, and do not have any components in common.

1.1 Projective space

If you have encountered Bézout's theorem before, you have probably seen a more precise version. The first way we can improve the statement is to consider the asymptotes more carefully.

Consider $g(x, y) = xy - 1$ and $f(x, y) = y - 1$. Then C has degree 1 and D has degree 2, so we expect their intersection to consist of two points. When we replaced C with a nearby curve C_s, this was indeed the case, but as $s \to 0$, one of those intersection points escaped to infinity *and was replaced by a common asymptote.* The asymptote really wants to be an intersection point!

If we count asymptotes as intersection points, maybe we can get a better version of Bézout's theorem. Unfortunatley, this isn't quite right: Consider $g(x, y) = xy - 1$ and $f(x, y) = y$. This time there are no intersection points at all, but moving C slightly we see that *two* intersection points are escaping to the same asymptote. In fact, this means that C and D are *tangent* at infinity.

We can get a better sense of what is going on with a change of coordinates. Let $x_1 = x^{-1}$ and $y_1 = y/x$. Note that these coordinates don't make sense near $x = 0$, but they do make sense when x is very large. The asymptotic intersection occurs at $(x_1, y_1) = (1, 0)$.

In these coordinates, the equation for D is $y_1/x_1^2 - 1 = 0$ or, rearranging, $y_1 = x_1^2$. The equation for C is $y_1/x_1 = 0$, or just $y_1 = 0$ by rearranging. These two curves are indeed tangent.

Secretly, we are working in local coordinates on the projective plane. By definition, the projective plane $\mathbf{CP}^2$ consists of all 1-dimensional subspaces in the 3-dimensional complex vector space $\mathbf{C}^3$. For each point (x, y) of $\mathbf{C}^2$ we have a line in $\mathbf{C}^3$ spanned by the vector $(x, y, 1)$. Thus $\mathbf{C}^2$ is contained in $\mathbf{CP}^2$, but $\mathbf{CP}^2$ is bigger. If we let (x, y) approach infinity in $\mathbf{C}^2$, the corresponding point of $\mathbf{CP}^2$ approaches a legitimate limit. (In other words, $\mathbf{CP}^2$ is compact.)

Theorem 1.4 (Bézout's theorem in the projective plane)**.** *For most* projective *algebraic plane curves C and D, the intersection $C \cap D$ consists of* $\deg(C)\deg(D)$ *points.*

'Most curves' may be interpreted to include curves that share no components and no tangencies.

1.2 Multiplicities

The statement of Bézout's theorem can be improved even more. We've noticed that tangencies correspond to collisions of pairs intersection points. Higher order tangencies correspond to higher order collisions:

Exercise 1.5. Let C be defined by $y - x^3 = 0$ and let D be defined by $y = 0$. Show that the intersection $C \cap D$ is a single point, but that if D is deformed to a nearby line, there are 3 points of intersection.

We can see the tangency algebraically. If we intersect $y - x^3 = 0$ with $x = 0$, we get the equations $x = y = 0$. This reflects the fact that these two curves intersect transversally. On the other hand, intersecting with $y = 0$ gives $x^3 = y = 0$. This is a different equation that has the same solutions in $\mathbf{C}$ as $x = y = 0$. However, *it has different solutions in some rings that are not fields.* This precisely reflects the fact that one line is tangent (to second order) and the other is not.

In the theory of schemes, not all points are treated equally. The equation $y = x^3 = 0$ defines a fatter point than does $y = x = 0$.

Exercise 1.6. (i) Show that $y = x = 0$ and $y = x^3 = 0$ have the same set of solutions in any field.

(ii) Find a commutative ring A such that $y = x = 0$ and $y = x^3 = 0$ have different solution sets in A.

Theorem 1.7 (Bézout's theorem with multiplicities). *For most* projective *algebraic plane curves C and D, the intersection $C \cap D$ consists of* $\deg(C)\deg(D)$ *points* when counted with multiplicity.

'Most curves' may be interpreted to include curves that share no components.

1.3 Intersection theory or derived algebraic geometry

The key in all of this discussion has been to consider moving our curves slightly. Intersection theory and derived algebraic geometry build this into the definition of intersection, yielding a very clean statement:

Theorem 1.8 (Bézout's theorem in intersection theory). All *projective algebraic plane curves C and D intersect in* $\deg(C)\deg(D)$ *points, provided the intersection is interpreted via intersection theory.*

1.4 Some more enumerative questions

Question 1.9. If $L_1, \dots, L_4$ are four lines in $\mathbf{C}^3$ how many lines L meet all four of them?

Question 1.10. If X is a surface in $\mathbf{C}^3$ defined by a cubic polynomial, how many lines lie on X?

2 A dictionary between algebra and geometry

In this section, we are going to investigate how geometric concepts are manifested algebraically and vice versa.

2.1 Points and functions

The most basic algebraic object we have at our disposal is an element of the ring $\mathbf{C}[x_1, \dots, x_n]$. We regard these as functions from $\mathbf{C}^n$ to $\mathbf{C}$.

The most basic geometric concept is that of a point. If $\xi \in \mathbf{C}^n$ is a point then we obtain a homomorphism

$$\begin{gathered} \mathrm{ev}_\xi : \mathbf{C}[x_1, \dots, x_n] \to \mathbf{C} \\ \mathrm{ev}_\xi(f) = f(\xi). \end{gathered}$$

Exercise 2.1 (Easy). Verify that this actually is a homomorphism and that it is surjective.

Since ev_ξ is surjective, its kernel is a maximal ideal, $\mathfrak{m}_\xi$. Hilbert's Nullstellensatz says that these are the only maximal ideals of $\mathbf{C}[x_1, \dots, x_n]$:

Theorem 2.2 (Corollary to Hilbert's Nullstellensatz). *Every maximal ideal of* $\mathbf{C}[x_1, \dots, x_n]$ *is of the form $(x_1 - \xi_1, \dots, x_n - \xi_n)$ for some $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n$.*

2.2 Algebraic subsets

Definition 2.3. If $J \subset \mathbf{C}[x_1, \ldots, x_n]$ is a set of polynomials then $V(J)$ is the set of all $\xi \in \mathbf{C}^n$ such that $f(\xi) = 0$ for all $f \in J$. An *algebraic subset* of $\mathbf{C}^n$ is a subset that is equal to $V(J)$ for some set J.

If X is a subset of $\mathbf{C}^n$, we define $I(X)$ to be the set of all $f \in \mathbf{C}[x_1, \ldots, x_n]$ such that $f(\xi) = 0$ for all $\xi \in X$.

Exercise 2.4 (A commutative algebra warmup)**.** (i) Let J' be the radical ideal generated by J. Show that $V(J) = V(J')$.[1]

(ii) For any $X \subset \mathbf{C}^n$ show that $I(X)$ is a radical ideal of $\mathbf{C}[x_1, \ldots, x_n]$.

Theorem 2.5 (Hilbert's Nullstellensatz)**.** *If $J \subset K[x_1, \ldots, x_n]$ is an ideal then $I(V(J)) = \sqrt{J}$.*

Exercise 2.6 (Easy, given the previous exercise)**.** Use the Nullstellensatz to prove that $V(I(X)) = X$ for any algebraic subset of $\mathbf{C}^n$.

Exercise 2.7 (Easy, given the previous exercise)**.** Give a one-to-one correspondence between algebraic subsets of $\mathbf{C}^n$ and (isomorphism classes of) surjections $\mathbf{C}[x_1, \ldots, x_n] \to A$ with A reduced.[2]

2.3 Morphisms of algebraic subsets

We already know what an algebraic function from $\mathbf{C}^n$ to $\mathbf{C}$ is: It's just a polynomial in the variables $x_1, \ldots, x_n$. In other words, it's an element of $\mathbf{C}[x_1, \ldots, x_n]$. If X is an algebraic subset of $\mathbf{C}^n$ then we declare that a morphism from X to $\mathbf{C}$ is an element of $\mathbf{C}[x_1, \ldots, x_n]$ with f and g considered equivalent if $f(\xi) = g(\xi)$ for all $\xi \in \mathbf{C}^n$.

Exercise 2.8. (i) Show that the morphisms from an algebraic set $X \subset \mathbf{C}^n$ to $\mathbf{C}$ are in canonical bijection with $\mathbf{C}[x_1, \ldots, x_n]/I(X)$.

(ii) Show that the maximal ideals of $\mathbf{C}[x_1, \ldots, x_n]/I(X)$ are the same as the maximal ideals of $\mathbf{C}[x_1, \ldots, x_n]$ that contain $I(X)$ are the same as the points of X. (Hint: You will want to use the Nullstellensatz (Theorem 2.2) here.)

Suppose $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^m$ are algebraic subsets. What is a morphism $X \to Y$? We should certainly have a morphism $X \to \mathbf{C}^m$ in this case, which amounts to m morphisms from X to $\mathbf{C}$. That is, it means we have m elements of $A = \mathbf{C}[x_1, \ldots, x_n]/I(X)$, which we can also regard as a homomorphism

$$\mathbf{C}[y_1, \ldots, y_m] \to A.$$

Notice that the left side is the set of algebraic functions on $\mathbf{C}^m$ and the right side is the set of algebraic functions on X. If φ denotes the map $X \to \mathbf{C}^m$ then this homomorphism just sends a function $f \in \mathbf{C}[y_1, \ldots, y_m]$ to $f \circ \varphi \in A$. We usually write $\varphi^* f$ for the function $f \circ \varphi$.

What does it mean for the image of φ to lie inside Y? It means that for any $f \in I(Y)$ and $\xi \in X$ we have $f(\varphi(\xi)) = 0$. In other words, $\varphi^* f(\xi) = 0$ for all $\xi \in X$. If g is

[1] An ideal J' is called *radical* if $f^n \in J' \implies f \in J'$.

[2] A commutative ring is *reduced* if it has no nonzero nilpotent elements.

a representative for $\varphi^* f$ in $\mathbf{C}[x_1, \ldots, x_n]$ then this means $g \in I(X)$. Thus $\varphi^* f = 0$ in $A = \mathbf{C}[x_1, \ldots, x_n]/I(X)$.

Thus our condition that φ define a map $X \to Y$ is that $\varphi^* I(Y) = 0$ in A. By the universal property of the quotient ring, this means φ^* can be regarded as a homomorphism

$$B = \mathbf{C}[y_1, \ldots, y_m]/I(Y) \to \mathbf{C}[x_1, \ldots, x_n]/I(X) = A.$$

Of course, this isn't surprising when we think about B as the ring of functions on Y. If we have a map $\varphi : X \to Y$ and f is a function on Y then $f \circ \varphi$ is a function on X.

2.4 Abstract algebraic sets

In the last section, we saw that every algebraic set $X \subset \mathbf{C}^n$ gave rise to a reduced, finite type $\mathbf{C}$-algebra. Conversely, every reduced, finite type $\mathbf{C}$-algebra is the quotient of some $\mathbf{C}[x_1, \ldots, x_n]$ by a radical ideal, hence corresponds to an algebraic set. Different choices of generators give different embeddings in $\mathbf{C}[x_1, \ldots, x_n]$ give different embeddings in $\mathbf{C}^n$, but the different algebraic sets are all isomorphic, according to our definition of morphisms of algebraic sets above.

Exercise 2.9. Show that there is a contravariant equivalence between algebraic sets and reduced[3] finite type[4] $\mathbf{C}$-algebras.

2.5 Tangent vectors

Exercise 2.10. (i) Let $\xi \in \mathbf{C}$. Construct an identification $\mathbf{C}[x]/\mathfrak{m}_\xi^2 \simeq \mathbf{C}[\epsilon]/(\epsilon^2)$ sending $x - \xi$ to ϵ.

(ii) Show that under this identification, the map

$$\mathbf{C}[x] \to \mathbf{C}[x]/\mathfrak{m}_\xi \simeq \mathbf{C}[\epsilon]/(\epsilon^2)$$

sends $f \in \mathbf{C}[x]$ to $f(\xi) + \epsilon f'(\xi)$. (Suggestion for how to think about this: Interpret x as $\xi + \epsilon$ and think about the Taylor series.)

Exercise 2.11. (i) Show that for any tangent vector v at a point $\xi \in \mathbf{C}^n$ the function

$$\delta : \mathbf{C}[x_1, \ldots, x_n] \to \mathbf{C}[\epsilon]/(\epsilon^2)$$
$$\delta(f) = f(\xi) + \big(v \cdot \nabla f(\xi)\big)\,\epsilon$$

is a homomorphism (∇f denotes the gradient).

(ii) Show that every tangent vector arises this way for a unique homomorphism. (Hint: Write $\delta(f) = \varphi_0(f) + \epsilon\varphi_1(f)$. Set $\xi_i = \varphi(x_i)$ to get the point. Set $v_i = \varphi_1(x_i)$ to get the vector.)

The next two exercises are not recommended. A lot of subtleties arise.

Exercise 2.12. Generalize the previous exercise to give an identification

$$TX \simeq \mathrm{Hom}_{\mathbf{C\text{-}Alg}}(\mathbf{C}[x_1, \ldots, x_n]/I(X), \mathbf{C}[\epsilon]/(\epsilon^2))$$

when $X = V(J) \subset \mathbf{C}^n$ is a manifold.[5]

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[3] This means 'has no nilpotents'.

[4] This means 'finitely generated as a commutative ring'.

[5] This identification is *always* true once one has defined the tangent space of a singular space. We will later take this as the definition of the tangent space.

Geometry	Algebra
$\xi \in \mathbf{C}^n$	$\mathrm{ev}_\xi : \mathbf{C}[x_1, \ldots, x_n] \to \mathbf{C}$ $\mathfrak{m}_\xi \subset \mathbf{C}[x_1, \ldots, x_n]$ maximal ideal
$f : \mathbf{C}^n \to \mathbf{C}$	$f \in \mathbf{C}[x_1, \ldots, x_n]$ $\mathbf{C}[y] \to \mathbf{C}[x_1, \ldots, x_n]$
$X \subset \mathbf{C}^n$ algebraic subset	$I \subset \mathbf{C}[x_1, \ldots, x_n]$ radical ideal $\mathbf{C}[x_1, \ldots, x_n] \to A$ surjective, A reduced, finite type $\mathbf{C}$-algebra
algebraic set X $X = \mathrm{Hom}(A, \mathbf{C})$	reduced, finite type $\mathbf{C}$-algebra $A = \mathrm{Hom}(X, \mathbf{C})$
morphism of algebraic sets $f : X \to Y$	homomorphism of commutative rings $B \to A$
tangent vector $(\xi, v) \in TX$	$v \in (\mathfrak{m}_\xi/\mathfrak{m}_\xi^2)^\vee$ $A \to \mathbf{C}[\epsilon]/(\epsilon^2)$
affine scheme X	commutative ring A
morphism of affine schemes $X \to Y$	morphism of commutative rings $B \to A$

Exercise 2.13. Show that when X is a *compact* complex manifold,

$$TX \simeq \mathrm{Hom}_{\mathbf{C}\text{-}\mathbf{Alg}}(C^\infty(X), \mathbf{C}[\epsilon]/(\epsilon^2)).$$

A similar statement holds for real manifolds.

So far, we have shown that reduced, finite type $\mathbf{C}$-algebras correspond to algebraic sets. The algebra $\mathbf{C}[\epsilon]/(\epsilon^2)$ is not reduced, but if we broaden our horizons just a little and pretend it corresponds to a space D then we have

$$TX = \mathrm{Hom}(D, X).$$

That is D *is the universal point with tangent vector*! It will turn out that D is a scheme that really does consist of one point with a little bit of infinitesimal 'fuzz' around it.

Theorem 2.14 (Nullstellensätze)**.** *(i) (Zariski's lemma) If K is a field and L is a field extension of K that is finitely generated as a commutative ring then L is finite dimensional over K.*

(ii) (Weak Nullstellensatz) Let K be an algebraically closed field. Let $J \subset K[x_1, \ldots, x_n]$ be an ideal. Then $V(J) = \varnothing$ if and only if $\sqrt{J} = K[x_1, \ldots, x_n]$.

(iii) (Hilbert's Nullstellensatz) If $J \subset K[x_1, \ldots, x_n]$ is an ideal then $I(V(J)) = \sqrt{J}$.

(iv) If K is an algebraically closed field then the maximal ideals of $K[x_1, \ldots, x_n]$ are all of the form $(x_1 - \xi_1, \ldots, x_n - \xi_n)$ for $\xi_i \in K$.

Exercise 2.15 (Some parts of this exercise are likely to be hard)**.** Show that the different statements of Hilbert's Nullstellensatz are equivalent.

We won't prove the Nullstellensatz for a while. The modern perspective on algebraic geometry treats all prime ideals as points, not just the maximal ideals, so the Nullstellensatz isn't quite as fundamental. The Nullstellensatz for prime ideals is much easier, and we will prove it in the next lecture.

3 The prime spectrum and the Zariski topology

Reading 3.1. [Vak14, §§3.2–3.5, 3.7], [Mum99, §II.1], [AM69, Chapter 1, Exercises 15–28], [Har77, pp. 69–70]

3.1 The Zariski topology

In the last section, we saw that the *points* of $\mathbf{C}^n$ can be recovered algebraically from the ring $\mathbf{C}[x_1, \ldots, x_n]$. However, the *topology* of $\mathbf{C}^n$ can't be recovered algebraically. In this section we will see how to get a topology that is coarser than the usual topology on $\mathbf{C}^n$. The construction is quite general, and works with $\mathbf{C}[x_1, \ldots, x_n]$ replaced by any commutative ring.

Definition 3.2 (The prime spectrum)**.** We write $\operatorname{Spec} A$ for the set of prime ideals of a commutative ring A. For each $\mathfrak{p} \in \operatorname{Spec} A$, let

$$\mathbf{k}(\mathfrak{p}) = \operatorname{frac}(A/\mathfrak{p}).$$

This is called the *residue field* of $\mathfrak{p}$. We define

$$\operatorname{ev}_{\mathfrak{p}} : A \to A/\mathfrak{p} \to \operatorname{frac}(A/\mathfrak{p}) = \mathbf{k}(\mathfrak{p})$$

be the homomorphism that sends f to $f \bmod \mathfrak{p}$. It is convenient to write $f(\mathfrak{p})$ instead of $\operatorname{ev}_{\mathfrak{p}}(f)$, although one must take care to remember that $f(\mathfrak{p})$ and $f(\mathfrak{q})$ don't always live in the same set when $\mathfrak{p} \neq \mathfrak{q}$.

For any $J \subset A$, let

$$V(J) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \operatorname{ev}_{\mathfrak{p}}(J) = 0\}.$$

Equivalently, $V(J)$ is the set of $\mathfrak{p} \in \operatorname{Spec} A$ such that $J \subset \mathfrak{p}$.

We write $D(J)$ for the complement of $V(J)$ in $\operatorname{Spec} A$. When we need to emphasize the ring A, we write I_A, V_A, D_A, etc. When J consists of just one element f, we write $V(f) = V(\{f\})$ and $D(f) = D(\{f\})$.

A subset $Z \subset \operatorname{Spec} A$ is called *closed* if $Z = V(J)$ for some $J \subset A$. A subset $U \subset \operatorname{Spec} A$ is called *open* if $U = D(J)$ for some ideal J. Sets of the form $D(f)$ for $f \in A$ are called *principal open subsets* or *distinguished open subsets.*

We also define

$$I(Z) = \{f \in A \mid f(Z) = 0\} = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}.$$

Question 3.3. Here is something to think about: Is every open subset principal? We will answer this question later.

Exercise 3.4. Let $\mathfrak{p}$ be a prime ideal of A. Show that $\{\mathfrak{p}\}$ is a closed subset of $\operatorname{Spec} A$ if and only if $\mathfrak{p}$ is a maximal ideal.

The exercise shows that this topology usually is not Hausdorff. It contains many points that are not closed. This sounds pathological, but it turns out to be convenient once you get used to it.

Exercise 3.5. Show that

$$V(\sum J_i) = \bigcap V(J_i)$$
$$V(JK) = V(J) \cup V(K)$$

for any ideals J and K. Conclude that the definitions of open and closed sets in Definition 3.2 give a topology, called the *Zariski topology*.

3.2 Examples

Here are some useful facts from commutative algebra that you may want to recall for the following exercise and later ones:

Theorem 3.6. *(i) A principal ideal domain is a unique factorization domain.*

(ii) If A is a unique factorization domain then $A[x]$ is a unique factorization domain.

(iii) In a unique factorization domain, the ideal generated by an irreducible element is prime.

Exercise 3.7. Suppose k is a field.

(i) Show that $\operatorname{Spec} k$ is a single point.

(ii) Show that $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ is a single point.

(iii) Show that $\operatorname{Spec}(k \times k)$ consists of two points. What is the topology?

Exercise 3.8. Describe the points and topology of $\operatorname{Spec} \mathbf{Z}$.

Exercise 3.9. (i) Describe the points and topology of $\operatorname{Spec} \mathbf{C}[x]$.

(ii) Describe the points and topology of $\operatorname{Spec} \mathbf{R}[x]$.

(iii) Describe the points and topology of $\operatorname{Spec} \mathbf{Q}[x]$.

Exercise 3.10. Suppose A is a domain. Show that $\operatorname{Spec} A$ contains a point that is dense. This is called the *generic point* of $\operatorname{Spec} A$.

Exercise 3.11. Give a point of $\operatorname{Spec} \mathbf{C}[x, y]$ that is neither a point of $\mathbf{C}^2$ nor the generic point.

3.3 Basic properties

Exercise 3.12. Suppose that A is a commutative ring and $f \in A$. Show that there is a universal homomorphism $\varphi : A \to B$ such that $\varphi(f)$ is invertible. (Hint: $B = A[u]/(uf - 1)$.)

Exercise 3.13. (i) Show that $D(J) = \bigcup_{f \in J} D(f)$.

(ii) Show that the intersection of two principal open subsets is a principal open subset.

(iii) Conclude that the principal open subsets of $\operatorname{Spec} A$ form a basis of the Zariski topology.

Exercise 3.14 (Unions of principal open affine subsets. Important!)**.** (i) Suppose A is a commutative ring and $f_1, \ldots, f_n \in A$. Show that $\operatorname{Spec} A = \bigcup D(f_i)$ if and only if $(f_1, \ldots, f_n)A = A$.

(ii) Conclude that $\operatorname{Spec} A$ is *quasicompact*[6] for any commutative ring A.

Exercise 3.15 (The prime Nullstellensatz)**.** This is much easier than Hilbert's Nullstellensatz, which might also be called the *maximal Nullstellensatz*.

(i) (You may want to use this one to prove the next one, or you may want to skip this part because it is a special case of the next one.) For any commutative ring A, show that $I_A(\operatorname{Spec} A)$ is the radical of A. (Hint: Let f be an element of A that is contained in every prime ideal. Consider $A[f^{-1}]$. What are its prime ideals?)

(ii) For any commutative ring A and any subset $J \subset A$, show that $I(V(J))$ is the radical ideal generated by J. (Hint: Reduce to the previous part by replacing A with A/J. Or imitate the proof of the previous part.)

(iii) Conclude that $Z \subset \operatorname{Spec} A$ is closed if and only if $Z = V(I(Z))$.

Residue fields

We give two categorical characterizations of the points of the prime spectrum.

Exercise 3.16 (Minimal homomorphisms to fields)**.** Call a homomorphism from A to a field k *minimal* if, whenever L and K are fields and there is a commutative diagram of solid lines

$$\begin{tikzcd}
& k \arrow[dr] \arrow[dd, dashed] & \\
A \arrow[ur] \arrow[dr] & & L \\
& K \arrow[ur] &
\end{tikzcd}$$

there is a unique dashed arrow extending the diagram. Two homomorphisms $f : A \to k$ and $g : A \to k'$ are said to be *isomorphic* if there is an isomorphism $h : k \to k'$ with $hf = g$.

Show that the points of $\operatorname{Spec} A$ correspond to isomorphism classes of *minimal homomorphisms* $A \to k$, where k is a field.

Exercise 3.17 (Epimorphisms to fields)**.** A morphism of commutative rings $f : A \to B$ is called an *epimorphism* if, for any commutative ring C, composition with f induces an injection $\operatorname{Hom}(B, C) \to \operatorname{Hom}(A, C)$. In other words, f is an epimorphism if, for any homomorphisms $g, h : B \to C$, we have $gf = hf$ if and only if $g = h$.

(i) Show that any surjective homomorphism is an epimorphism.

(ii) Not every epimorphism is a surjection: Suppose that A is an integral domain and B is its field of fractions. Show that $A \to B$ is an epimorphism. (More generally, show that any localization is an epimorphism.)

[6] This is what people in North America usually call compact. It means that every open cover of $\operatorname{Spec} A$ has a finite subcover.

(iii) (This part may be difficult. The proof suggested below works in much greater generality, and we will see it repeatedly.) Show that a homomorphism from a commutative ring A to a field K is an epimorphism if and only if the image of A generates K as a field. (Hint: Replace A with the field generated by its image in k. Show that $k \to K$ is an epimorphism if and only if $k = K$ by taking $C = K \otimes_A K$ in the definition of an epimorphism. Let $i, j : K \to K \otimes_k K$ be given by $i(x) = 1 \otimes x$ adn $j(x) = x \otimes 1$. Prove that $i = j$ if and only if $k = K$ by proving the sequence

$$0 \to k \to K \xrightarrow{i-j} K \otimes_k K$$

is exact. Do this by proving the sequence

$$0 \to K \to K \otimes_k K \xrightarrow{i'-j'} K \otimes_k K \otimes_k K$$

with $i'(y \otimes x) = y \otimes 1 \otimes x$ and $j'(y \otimes x) = y \otimes x \otimes 1$ is exact. To verify this, show that the maps $K \otimes_k K \to K$ sending $y \otimes x$ to yx and $K \otimes_k K \otimes_k K \to K \otimes_k K$ sending $z \otimes y \otimes x$ to $zy \otimes x$ split the sequence.

3.4 Functoriality

Exercise 3.18 (Functoriality of the prime spectrum)**.** Suppose $f : A \to B$ is a homomorphism of commutative rings. Show that $p \mapsto f^{-1}p$ defines a function $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$. Show that this definition respects composition of homomorphisms.

Exercise 3.19. Let $\varphi : A \to B$ be a homomorphism and let $u : \operatorname{Spec} B \to \operatorname{Spec} A$ by the induced morphism of spectra. Show that $u^{-1}D(f) = D(\varphi(f))$.

Exercise 3.20. Show that every point of $\operatorname{Spec} A$ corresponds to a homomorphism of commutative rings $A \to k$ for some field k.

Exercise 3.21 (The universal property of an open subset)**.** Let $J \subset A$ be any subset and let $D(J) \subset \operatorname{Spec} A$ be an open subset. Show that the map $\operatorname{Spec} B \to \operatorname{Spec} A$ associated to $f : A \to B$ factors through $D(J)$ if and only if $f(J)B = B$.

Exercise 3.22 (The universal property of a closed subset)**.** Show that $\varphi : A \to B$ induces a map $g : \operatorname{Spec} B \to \operatorname{Spec} A$ factoring through $V(J)$ if and only if $f(J)B$ is a nilideal (every element is nilpotent).

These two exercises will be generalized later, so there is no need to do them now except to build intuition.

Exercise 3.23 (Fibers of morphisms of spectra)**.** Suppose that $A \to B$ is a homomorphism of commutative rings and $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is the corresponding map on spectra. Let p be a point of $\operatorname{Spec} A$ and let $A \to k$ be the corresponding homomorphism from A to a field k. Identify $f^{-1}p$ with $\operatorname{Spec}(B \otimes_A k)$. (Hint: A homomorphism from B to a field K such that $A \to B \to K$ factors through k is precisely the same as a homomorphism $B \otimes_A k \to K$.)

Exercise 3.24 (Surjectivity of integral morphisms)**.** Suppose A is a commutative ring and f is an integral polynomial with coefficients in A. Let $B = A[t]/(f)$. Show that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. (Hint: Reduce to the case where A is a field.)

3.5 More examples

Exercise 3.25. If R is a discrete valuation ring then $\operatorname{Spec} R$ consists of two points, one open and dense and the other closed. (If you don't know what a discrete valuation ring is, assume $R = \mathbf{C}[t]_{(t)}$ or $R = \mathbf{Z}_{(p)}$.)

Exercise 3.26. Describe the points and topology of $\mathbf{C}[x, y]_{(x,y)}$.

Exercise 3.27. Describe the points and topology of $\mathbf{Z}[x]$.

Exercise 3.28. Let $A' \to A$ be a surjection of commutative rings whose kernel is nilpotent. Show that the map $\operatorname{Spec} A \to \operatorname{Spec} A'$ is a homeomorphism.

Exercise 3.29. Suppose A is a commutative ring.

(i) Let $\mathfrak{p}$ be a prime ideal of A. Show that $\{\mathfrak{p}\}$ is dense in $V(\mathfrak{p})$. Conclude that $V(\mathfrak{p})$ is *irreducible*: it is impossible to write $V(\mathfrak{p})$ as the union of two closed subsets $A \cup B$ unless at least one of them is equal to $V(\mathfrak{p})$ itself.

(ii) Suppose that $Z \subset \operatorname{Spec} A$ is an irreducible subset. Show that there is a unique prime ideal $\mathfrak{p} \subset A$ such that $V(\mathfrak{p}) = Z$.

Chapter 2

Introduction to schemes

4 Sheaves I

Reading 4.1. [Vak14, §§2.1–2.4, 2.7 (pp. 69–83)], [Har77, §II.1 (pp. 60–65)]

4.1 Why sheaves?

In geometry, one usually has a *ring of functions* associated to a space. For example, in differential geometry one can take the ring of C^∞ functions, valued in $\mathbf{R}$ or in $\mathbf{C}$. In topology, one has a ring of continuous functions, valued in $\mathbf{R}$ or $\mathbf{C}$ (or any topological ring).

In algebraic geometry, we turn this around and declare that every commutative ring *should be* the ring of functions on some space, which we call an affine scheme. We also allow ourselves to glue spaces together along open subsets. In a sense, schemes are the minimal collection of spaces that can be constructed from these axioms.

It is possible to proceed quite formally along these lines, and we will discuss this in Lecture **??**. For the sake of concreteness, and adherence to historical conventions, we will first give a definition in which schemes do have an underlying space. However, there will be one very strange departure: functions are not determined by their values at points.

Exercise 4.2. Give an example of a commutative ring A and two elements $f, g \in A$ such that $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(g)$ for all $\xi \in \operatorname{Spec} A$. Interpret this as the failure of functions to be determined by their values at points.

In differential geometry, for example, one can describe the maps of differentiable manifolds $X \to Y$ as the functions on the underlying sets that have some desirable local property. Since functions in algebraic geometry are not determined by their values at points, one cannot specify morphisms between schemes this way. Instead, we need to explicitly specify the ring of functions. We describe a morphism of schemes as a morphism of topological spaces *with compatible homomorphism between their rings of functions.*

But schemes can be glued together from affine schemes in nontrivial ways. In contrast to differentiable manifolds, where the global functions always determine the local functions, schemes often do not have many global functions at all. In fact, one already sees this in complex geometry:

Exercise 4.3. Show that all holomorphic functions from a compact Riemann surface to $\mathbf{C}$ are constant.

Locally, a scheme may have a lot of functions, but these can fail to glue together to give global functions. When thinking about functions on a scheme, one is therefore obliged to think about functions on *all open subsets simultaneously*. In other words, one thinks about the *sheaf* of functions, not just the sheaf's ring of global sections.

4.2 Definitions

Definition 4.4. Let X be a topological space. A *presheaf (of sets)* on X consists of the following data and conditions:

PSH1 a set $F(U)$ for each open $U \subset X$ (one often writes $\Gamma(U, F) = F(U)$);

PSH2 a function $\rho_{UV} : F(U) \to F(V)$ whenever $V \subset U$ are open subsets of X;

PSH3 equality $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ when $W \subset V \subset U$ are open subsets of X.

A presheaf is called a *sheaf* if it satisfies the following additional conditions:

SH1 if $\xi, \eta \in F(U)$ and $\bigcup U_i = U$ and $\xi\big|_{U_i} = \eta\big|_{U_i}$ for all i then $\xi = \eta$;

SH2 if $\bigcup U_i = U$ and $\xi_i \in F(U_i)$ for all i and $\xi_i\big|_{U_i \cap U_j} = \xi_j\big|_{U_i \cap U_j}$ then there is a $\xi \in F(U)$ such that $\xi\big|_{U_i} = \xi_i$ for all i.

We obtain sheaves of groups, abelian groups, rings, commutative rings, etc. by substituting the appropriate concept for set and the appropriate notion of homomorphism for function in the definition of a presheaf.

Exercise 4.5. (i) All presheaves on a point are sheaves.

(ii) The category of sheaves on a point is equivalent (in fact isomorphic) to the category of sets.

Exercise 4.6. Suppose F is a sheaf on a topological space. Prove that $F(\varnothing)$ is a 1-element set.

4.3 Examples of sheaves

Exercise 4.7 (Constant presheaf)**.** Let X be a topological space and let S be a set with at least 2 elements. Define $F(U) = S$ for all open $U \subset X$. Give an example of X for which F is not a sheaf. (Hint: Exercise 4.9 below may give a hint.) For almost any space X you pick, F will not be a sheaf, but try to find a simple example.

Exercise 4.8 (Subsheaves)**.** If F and G are presheaves, we say that F is a *subpresheaf* of G if $F(U) \subset G(U)$ for all U. Suppose that G is a sheaf and F is a subpresheaf of G. Prove that F is a sheaf if and only if whenever $\xi \in G(U)$ and there is an open cover of U by sets V such that $\xi\big|_V \in F(V)$ then $\xi \in F(U)$.

Exercise 4.9 (Sheaf of functions)**.** You should do at least one of this exercise or the next.

Let X and Y be topological spaces and define $F(U)$ to be the set of continuous functions $U \to Y$, for each $U \in \mathbf{Open}(X)$. Show that F is a sheaf.

The collection of all functions is also a sheaf. If X and Y are manifolds, the differentiable functions form a sheaf.

This exercise generalizes the previous one. It will be important later, but it is not essential to do it now. Later we will see that every sheaf arises from the following construction!

Exercise 4.10 (Sheaf of sections)**.**

Suppose that $\pi : E \to X$ is a continuous function. For each open $U \subset X$, let $F(U)$ be the set of continuous functions $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$. These are called *sections* of E over U. Prove that F is a sheaf.

4.4 Morphisms of sheaves

Definition 4.11. If F and G are presheaves on a topological space X, a *morphism* $\varphi : F \to G$ consists of functions $\varphi_U : F(U) \to G(U)$ such that whenever $V \subset U$ is an open subset we have $\rho_{UV} \circ \varphi_U = \varphi_V \circ \rho_{UV}$.

A morphism of sheaves is a morphism of the underlying presheaves.

⋆ **Exercise 4.12** (The sheaf of morphisms)**.** Suppose that F and G are presheaves on X. For each open $U \subset X$, let $H(U) = \mathrm{Hom}_{\mathbf{Sh}(U)}(F\big|_U, G\big|_U)$.

(i) Show that H is a presheaf in a natural way.

(ii) Show that if G is a sheaf then H is a sheaf.

4.5 Sheaves are like sets

Virtually any definition concerning sets can be interpreted in $\mathbf{Sh}(X)$ if we interpret $\forall$ and $\exists$ as follows:

(i) $\forall\, \xi \in F$ means "for all open U and all $\xi \in F(U)\ldots$"

(ii) $\exists\, \xi \in F$ means "there is an open cover U_i and $\xi_i \in F(U_i)\ldots$

For example:

Definition 4.13. A morphism of sets $\varphi : F \to G$ is injective if for all $\xi, \eta \in F$ we have $\varphi(\xi) = \varphi(\eta)$ only if $\xi = \eta$. A morphism of sheaves $\varphi : F \to G$ is *injective* if, for all open subsets U of X and all $\xi, \eta \in F(U)$, we have $\varphi(\xi) = \varphi(\eta)$ only if $\xi = \eta$.

A morphism of sets $\varphi : F \to G$ is surjective if for all $\eta \in G$ there is some $\xi \in F$ such that $\varphi(\xi) = \eta$. A morphism of sheaves of sets $\varphi : F \to G$ is *surjective* if, for all open $U \subset X$ and all $\eta \in G(U)$, there is an open cover $U = \bigcup V_i$ and elements $\xi_i \in F(V_i)$ such that $\varphi(\xi_i) = \eta\big|_{V_i}$.

Exercise 4.14. The axiom of choice says that for every surjection $\varphi : F \to G$ there is a morphism $\sigma : G \to F$ such that $\varphi \circ \sigma = \mathrm{id}_G$. Show that the axiom of choice is false in $\mathbf{Sh}(S^1)$, where S^1 is the circle. (Hint: Let F be the sheaf of sections of the universal cover $\mathbf{R} \to S^1$ and let G be the final sheaf $G(U) = 1$ for all open $U \subset S^1$.)

Exercise 4.15. Show that a morphism of sheaves $\varphi : F \to G$ is an isomorphism if and only if it is *bijective* (both injective and surjective).

4.6 Sheaves on a basis

Reading 4.16. [Vak14, §2.7]

Suppose X is a topological space and $\mathscr{U} \subset \mathbf{Open}(X)$ is a basis of X. The definition of a presheaf on $\mathscr{U}$ is obtained by substituting $\mathscr{U}$ for $\mathbf{Open}(X)$ in Definition 4.4. The definition of a sheaf requires a small modification in condition **SH2**.

Definition 4.17. Let $\mathscr{U}$ be a basis for a topological space X. A presheaf on $\mathscr{U}$ is said to be a sheaf if it satisfies **SH1** and

SH2′ if $U = \bigcup_{i,j \in I} U_i$ is an open cover in $\mathscr{U}$, and $\xi \in F(U_i)$ are elements such that, for each $i, j \in I$, there is an open cover $U_i \cap U_j = \bigcup_{k \in K_{ij}} V_{ijk}$ with $\xi_i\big|_{V_{ijk}} = \xi_j\big|_{V_{ijk}}$ for all $k \in K_{ij}$, then there is a $\xi \in F(U)$ such that $\xi\big|_{U_i} = \xi_i$ for all $i \in I$.

Exercise 4.18. Show that if **SH1** holds for F and $\mathscr{U}$ is stable under finite intersections, conditions **SH2** and **SH2′** are equivalent.

Theorem 4.19. *Suppose $\mathscr{U} \subset \mathbf{Open}(X)$ is a basis. If F is a sheaf on $\mathscr{U}$ then F extends in a unique way (up to unique isomorphism) to a sheaf on $\mathbf{Open}(X)$.*

This procedure can be viewed as an example of *sheafification*, which we will discuss in Lecture 8 using the *espace étalé*.

For each $V \in \mathbf{Open}(X)$, define

$$G(V) = \varprojlim_{\substack{U \in \mathscr{U} \\ U \subset V}} F(U).$$

In case you are not familiar with limits, here is an explicit description of this limit: It is a tuple $(\xi_U)_{\substack{U \in \mathscr{U} \\ U \subset V}}$ with each $\xi_U \in F(U)$ such that whenever $U' \subset U$ we have $\xi_U\big|_{U'} = \xi_{U'}$.

Exercise 4.20. Construct a canonical isomorphism $G\big|_{\mathscr{U}} \simeq F$.

If H is any sheaf extending F to $\mathbf{Open}(X)$ then consider $\xi \in H(V)$. For every $U \subset V$ in $\mathscr{U}$, we have $\xi\big|_U \in F(U)$ and $\xi\big|_U\big|_{U \cap U'} = \xi\big|_{U'}\big|_{U \cap U'}$ so ξ determines an element of $G(V)$ (using **SH2**). This element is unique by **SH1**.

Exercise 4.21. Fill in the details from the last paragraph to show that if there is a sheaf H extending F then there is a unique isomorphism $H \to G$.

Exercise 4.22. Complete the proof by showing G is a sheaf.

Exercise 4.23. Show that a morphism of sheaves defined on a basis of open sets extends uniquely to a morphism defined on the whole space.

5 Ringed spaces and schemes

Reading 5.1. [Vak14, §§3.2, 3.4, 3.5, 4.1, 4.3], [Mum99, §II.1], [Har77, pp. 69–74]

Definition 5.2 (Ringed space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X.

We usually write X for a ringed space $(X, \mathcal{O}_X)$, effectively using the same symbol for both the underlying space and the space together with its structure sheaf. This is an abuse of terminology, but usually doesn't cause too much trouble. If we must distinguish X from its underlying topological space, we write $|X|$ for the topological space.

There are many familiar examples:

Not an important exercise to write up carefully.

Exercise 5.3. The following are ringed spaces:

(i) X is a manifold and $\mathcal{O}_X$ is the sheaf of C^∞ functions on X valued in $\mathbf{R}$ or $\mathbf{C}$;

(ii) X is a topological space and $\mathcal{O}_X$ is the sheaf of continuous functions on X, valued in any topological commutative ring;

(iii) X is a complex manifold and $\mathcal{O}_X$ is the sheaf of holomorphic functions on X;

This exercise is very important! It illustrates all kinds of useful techniques.

Exercise 5.4. Let A be a commutative ring. Define $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A[f^{-1}]$ for each principal open affine $D(f) \subset \operatorname{Spec} A$.

(i) Define the restriction homomorphisms in a natural way so that this is a presheaf of commutative rings on the basis of principal open subsets $\operatorname{Spec} A$.

(ii) Show that this presheaf is a sheaf.

Definition 5.5. A ringed space $(X, \mathcal{O}_X)$ is called an *affine scheme* if it is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some commutative ring A. A *scheme* is a ringed space that is locally an affine scheme.[1]

5.1 Descent

The following exercises guide you through one solution to Exercise (5.11) (ii).

Exercise 5.6. Let F be a presheaf on a basis $\mathscr{U}$ for a topological space X.

(i) Show that F satisfies **SH1** if and only if

$$F(U) \to \prod_{i \in I} F(U_i) \tag{5.1}$$

is injective whenever $U = \bigcup U_i$ is an open cover of U in $\mathscr{U}$.

(ii) Show that a particular instance of (5.1) is injective if and only if there is a subcollection $J \subset I$ such that $\bigcup_{i \in J} U_i = U$ and

$$F(U) \to \prod_{i \in J} F(U_i) \tag{5.2}$$

is injective.

Exercise 5.7. Assume that F is a separated presheaf (this means F satisfies **SH1**) on a basis $\mathscr{U}$ for a topological space X that is closed under intersections.

(i) Show that F satisfies **SH2** if and only if

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j) \tag{5.3}$$

is exact[2] whenever $U = \bigcup U_i$ is an open cover of U in $\mathscr{U}$. (Note: Make sure you understand what all of the maps are in this diagram!)

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[1] Warning: In older literature, what is today called a 'scheme' was called a 'prescheme'. The word 'scheme' was reserved for what is today called a 'separated scheme'.

[2] A diagram of sets $A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$ is said to be *exact* if f is injective and the image of f is exactly the collection of all $b \in B$ such that $g(b) = h(b)$. This condition is equivalent to exactness of the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g-h} C$ when the objects are abelian groups.

(ii) Show that a particular instance of (∗) is exact if and only if there is a subcollection $J \subset I$ such that $\bigcup_{i \in J} U_i = U$ and the sequence

$$F(U) \to \prod_{i \in J} F(U_i) \rightrightarrows \prod_{i,j \in J} F(U_i \cap U_j) \tag{5.4}$$

is exact.

Exercise 5.8. Combine the previous two exercises to show that a presheaf on a basis $\mathscr{U}$ of quasicompact open subsets that is closed under intersection is a sheaf if and only if the sequences (∗) are exact whenever $U = \bigcup U_i$ is a finite cover in $\mathscr{U}$.

Exercise 5.9. Generalize the last two exercises to apply to all bases, not just those closed under intersections.

Exercise 5.10. Let A be a commutative ring. Show that a sequence of A-modules

$$M' \to M \to M''$$

is exact if and only if the sequence of localized modules[3]

$$M'_f \to M_f \to M''_f$$

is exact.

Exercise 5.11. Prove that $\mathcal{O}_{\operatorname{Spec} A}$ is a sheaf on the basis $\mathscr{U}$ of principal open affine subsets of $\operatorname{Spec} A$:

(i) Reduce the problem to showing that the sequence

$$A \longrightarrow \prod_{i \in I} A[f_i^{-1}] \Longrightarrow \prod_{i,j \in I} A[f_i^{-1}, f_j^{-1}] \tag{5.5}$$

is exact whenever I is a finite subset of A such that $IA = A$.

Prove that exactness of this sequence is equivalent to exactness of the sequences

$$A_f \longrightarrow \prod_{i \in I} A_f[f_i^{-1}] \Longrightarrow \prod_{i,j \in I} A_f[f_i^{-1}, f_j^{-1}] \tag{5.6}$$

for all $f \in I$. (Note: $A_f = A[f^{-1}]$. The mixed notation is just to make the equation look prettier.) (Warning: Be careful about commuting localization with products.)

Prove that the sequences (5.6) are exact. (Hint: You can do this by explicitly splitting the sequence or by using a chain homotopy. There is a way to do this that doesn't require any messy algebra, using Exercises 5.6 and 5.7.)

[3]Recall that $M_f = A[f^{-1}] \otimes_A M$ is the module over $A[f^{-1}]$ induced by M. It can be constructed explicitly as the set of symbols $f^{-n}x$ with $x \in M$, subject to the relation $f^{-n}x = f^{-m}y$ if there is some k such that $f^k(f^m x - f^m y)$. It can also be constructed as the direct limit $\varinjlim_{n \in \mathbf{N}} f^{-n}M$, where $f^{-n}M = M$ for all M and the map $f^{-n}M \to f^{-m}M$ for $n < m$, sends $f^{-n}x \in f^{-n}M$ to $f^{-m}f^{m-n}x \in f^{-m}M$.

5.2 Partitions of unity

This section guides you through another solution to Exercise (5.11) (ii) that shares some spiritual similarity with the partition of unity arguments that appear in differential geometry.

Exercise 5.12. Reduce the problem to showing that the sequence

$$0 \to A \to \prod_{i \in I} A[f_i^{-1}] \to \prod_{i,j \in I} A[f_i^{-1}, f_j^{-1}] \tag{5.7}$$

is exact for whenever $I \subset A$ and $IA = A$. Make sure you know what the maps in this sequence are before you try to prove anything!

Exercise 5.13. (i) If $x \in A$ and x restricts to zero in $A[f_i^{-1}]$ then $f_i^n x = 0$ for some $n \geq 0$.

(ii) If $(f_1, \ldots, f_k) = A$ then $(f_1^{n_1}, \ldots, f_k^{n_k}) = A$ as well.

Exercise 5.14. (i) Prove the exactness of (5.7) at A.

(ii) Prove the exactness of (5.7) at $\prod_{i \in I} A[f_i^{-1}]$.

Chapter 3

First properties of schemes

6 Examples

Reading 6.1. [Vak14, §§4.4]

6.1 Open subschemes

Exercise 6.2. Show that an open subset of a scheme is equipped with the structure of a scheme in a natural way. (Hint: Restriction of a sheaf is a sheaf.)

6.2 Affine space

Definition 6.3 (Affine space)**.** The scheme $\operatorname{Spec} \mathbf{Z}[x_1, \ldots, x_n]$ is denoted $\mathbf{A}^n$ and is called *n-dimensional affine space.*

6.3 Gluing two affine schemes

Exercise 6.4. Suppose that X and Y are two schemes and $U \subset X$ and $V \subset Y$ are open subsets such that $U \simeq V$ and under this isomorphism $\mathcal{O}_U \simeq \mathcal{O}_V$. Construct a scheme Z whose underlying topological space is the union of X with Y along $U \simeq V$ and for which $\mathcal{O}_Z\big|_X = \mathcal{O}_X$ and $\mathcal{O}_Z\big|_Y = \mathcal{O}_Y$. (The statement of this exercise is deliberately vague in several ways. Part of your job is to make it precise.)

Exercise 6.5. Using the notation of the last exercise, let k be a field (you may find it easier to assume k is algebraically closed) and let $X = \mathbf{A}^1_k = \operatorname{Spec} k[x]$ and let $Y = \mathbf{A}^1_k = \operatorname{Spec} k[y]$. There is an open subset $U = D(x) \subset X$ and $V = D(y) \subset Y$.

(i) Construct two distinct homeomorphisms $U \simeq V$ and corresponding identifications $\mathcal{O}_U \simeq \mathcal{O}_V$. (Hint: One should correspond to $x = y$ and one should correspond to $x = y^{-1}$.)

(ii) Apply the last exercise to obtain a scheme Z for each of these two isomorphisms. Describe these spaces qualitatively and explain why they are different. (Hint: Consider a point $\xi \in X$. Move ξ so that $x(\xi)$ approaches 0. Move ξ so that $x(\xi)$ approaches ∞.)

6.4 Gluing more than two affine schemes

Exercise 6.6. How should this construction be modified when gluing 3 or more affine schemes along open subsets?

6.5 Projective space

In topology, $\mathbf{CP}^n$ is the set of 1-dimensional subspaces of $\mathbf{C}^{n+1}$. It is topologized as the quotient of $\mathbf{C}^{n+1} \smallsetminus \{0\}$ by $\mathbf{C}^*$. For each $i = 0, \dots, n$, let $V_i \subset \mathbf{C}^{n+1}$ be the span of the n coordinate vectors excluding e_i. Then $e_i + V_i$ consists of all vectors whose i-th coordinate is 1. The image of $e_i + V_i$ is an open subset U_i of $\mathbf{CP}^n$ and this gives a system of charts for $\mathbf{CP}^n$ as a complex manifold.

We can't imitate all of this algebraically, at least not yet. However, we can imitate the charts.

For each i, let $U_i = \operatorname{Spec} A_i$ where

$$A_i = \mathbf{Z}[x_{0/i}, \dots, x_{n/i}]/(x_{i/i} - 1).$$

The choice of notation makes the gluing that is about to happen easier. Later on, we will see that there is a way to think about $x_{k/i}$ as x_k/x_i in some bigger ring, but introducing this notation now would probably be misleading. Let

$$A_{ij} = A[x_{j/i}^{-1}].$$

Exercise 6.7. Verify that there is an identification $A_{ij} \simeq A_{ji}$ sending

$$x_{k/i} \mapsto x_{k/j} x_{i/j}^{-1}.$$

Let

$$A_{ijk} = A[x_{j/i}^{-1}, x_{k/i}^{-1}].$$

Note that $A_{ijk} = A_{ikj}$ and that the exercise above gives induces an isomorphism $A_{ijk} \to A_{jik}$ for any i, j, k.

Exercise 6.8. The previous exercise gives *two* identifications between A_{ijk} and A_{kji}:

$$A_{ijk} \to A_{jik} = A_{jki} \to A_{kji}$$
$$A_{ijk} = A_{ikj} \to A_{kij} = A_{kji}$$

Show that these two maps are the same.

Exercise 6.9. Use these identifications to glue the $U_i = \operatorname{Spec} A_i$ together into a scheme, $\mathbf{P}^n$.

Projective space is extremely important because almost every scheme one encounters in practice can be constructed as the intersection of an open subscheme and a closed subscheme of projective space.

A The homogeneous spectrum and projective schemes

Reading A.1. [Vak14, §4.5], [Har77, §I.2; §II.2, pp. 76–77]

A.1 Some geometric intuition

The exercises in this section are not required (and may not even be well-posed). The idea here is to get an idea of where the construction in the next section comes from. We will see how to make all of these ideas precise later when we talk about algebraic groups.

We limit attention to schemes over $\mathbf{C}$. Recall that $\mathbf{CP}^n$ is the quotient of $\mathbf{C}^{n+1} \smallsetminus \{0\}$ by $\mathbf{C}^*$.

Note that $\mathbf{C}^*$ acts on $\mathbf{C}^{n+1}$ by scaling the coordinates. How does this translate geometrically? If f is a function on $\mathbf{C}^{n+1}$ and $\lambda \in \mathbf{C}^*$ then we get a (right) action of $\mathbf{C}^*$ on f by defining $f.\lambda(x) = f(\lambda x)$. Note that this definition is a bit sloppy because a function on a scheme is not always determined by its values. In this case this turns out to be okay, since functions are determined by their values on $\operatorname{Spec} \mathbf{C}[x_0, \ldots, x_n]$, but we will have to wait until later to see how to make this construction make sense more generally.

In other words, $\mathbf{C}^*$ acts on the ring $\mathbf{C}[x_0, \ldots, x_n]$. Now, the actions $\mathbf{C}^*$ are well-understood. Any time $\mathbf{C}^*$ acts on a complex vector space V, we can decompose that vector space as a direct sum

$$V = \sum_{d \in \mathbf{Z}} V_d$$

where $\lambda \in \mathbf{C}^*$ acts on $v \in V_d$ by $\lambda.v = \lambda^d v$.

Correction! Thanks to Jon Lamar for noticing that the previous version of this exercise was incorrect.

Exercise A.2. Prove that every *algebraic* representation of $\mathbf{C}^*$ can be written as a direct sum as above. (Hint: Use the fact that commuting diagonalizable endomorphisms of a vector space can be diagonalized simultaneously and that $\mathbf{C}^*$ contains lots of roots of unity. Then show the only linear algebraic actions of $\mathbf{C}^*$ on $\mathbf{C}$ are by $\lambda.x = \lambda^d x$ for some $d \in \mathbf{Z}$.)

Thus there is a grading on $\mathbf{C}[x_0, \ldots, x_n]$.

Exercise A.3. Verify that this is the usual grading by degree.

Thus, at least morally speaking, graded $\mathbf{C}$-algebras correspond to affine schemes over $\mathbf{C}$ with an action of $\mathbf{C}^*$.

We want to take the quotient of $\mathbf{C}^{n+1}$ by this action. However, this has no hope of being a reasonable geometric space because it wants to have a dense point with residue field $\mathbf{C}$ corresponding to the orbit $\{0\}$. We only have a chance of getting something reasonable if we delete the fixed point.

So we hope that $P = (\mathbf{C}^{n+1} \smallsetminus \{0\})/\mathbf{C}^*$ will turn out to be a scheme. Let's try to find a reasonable open cover. The open subsets of P correspond to $\mathbf{C}^*$-invariant open subsets of $\mathbf{C}^{n+1} \smallsetminus \{0\}$.

Exercise A.4. Show that $D(f) \subset \mathbf{C}^{n+1}$ is $\mathbf{C}^*$-invariant if and only if f is a homogeneous polynomial or zero.

We write $D_+(f) \subset P$ for the open subset corresponding to $D(f) \subset \mathbf{C}^{n+1} \smallsetminus \{0\}$.

Now we figure out what $\mathcal{O}_P(D_+(f))$ should be. A function on $D_+(f)$ should be a function on $D(f)$ that is invariant under the action of $\mathbf{C}^*$. That is, we should have $f.\lambda = f$, which means precisely that f has graded degree zero. (Recall that f has graded degree d if $f.\lambda = \lambda^d f$.) The functions of with this property are exactly the ones of graded degree zero. Thus we get

$$\mathcal{O}_P(D_+(f)) = \mathbf{C}[x_0, \ldots, x_n, f^{-1}]_0.$$

How much of this can be generalized? What if we had any affine $\mathbf{C}$-scheme at all with an action of $\mathbf{C}^*$. This corresponds to a graded ring S. We could imitate the above procedure, but we have to delete the locus in $\operatorname{Spec} S$ that is fixed by $\mathbf{C}^*$.

Exercise A.5. Show that the fixed locus of $\mathbf{C}^*$ acting on $\operatorname{Spec} S$ is $V(S_{\neq 0})$.

So the first step is to delete $V(S_{\neq 0})$. We then attempt to take a quotient of $D(S_{\neq 0})$ by $\mathbf{C}^*$. Again, the $\mathbf{C}^*$-invariant open subsets are of the form $D(f)$ where f is homogeneous, and these correspond to open subsets $D_+(f) \subset P = D(S_{\neq 0})/\mathbf{C}^*$. We define $\mathcal{O}_P(D_+(f)) = \mathcal{O}_{\operatorname{Spec} S}(D(f))_0$.

A.2 The homogeneous spectrum of a graded ring

Definition A.6. A *graded ring* is a commutative ring S and a decomposition of the underlying abelian group of S is a direct sum: $S = \sum_{n \in \mathbf{Z}} S_n$ such that $S_n S_m \subset S_{n+m}$ for all $n, m \in \mathbf{Z}$. We write $S_{<0} = \sum_{n<0} S_n$, $S_{>0} = \sum_{n>0} S_n$, and $S_{\neq 0} = \sum_{n \neq 0} S_n$.[1]

[1] In most treatments, S is assumed to be non-negatively graded, i.e., $S_{<0} = 0$.

An element of S is called *homogeneous* if it is contained in some S_n, in which case it is said to have *degree* n. An ideal of S is called *homogeneous* if it is generated by homogeneous elements. A homomorphism of $\mathbf{Z}$-graded rings $f : S \to T$ is a homomorphism of rings such that $f(S_n) \subset T_n$ for all $n \in \mathbf{Z}$.

Correction! The irrelevant ideal is not merely the ideal generated by $S_{\neq 0}$: it is the radical ideal generated by $S_{\neq 0}$.

The radical ideal generated by $S_{\neq 0}$ is called the *irrelevant ideal* and is denoted S_+.

Exercise A.7. Show that if S contains a unit of non-zero degree then the irrelevant ideal is S itself.

Exercise A.8. (i) Show that an ideal $I \subset S$ is homogeneous if and only if $I = \sum (I \cap S_n)$.

(ii) [Vak14, Exercise 4.5.C (a)] Show that $I \subset S$ is a graded ideal if and only if it is the kernel of a homomorphism of graded rings.

(iii) [Vak14, Exercise 4.5.C (b)] Show that sums, products, intersections, and radicals of homogeneous ideals are homogeneous ideals.

(iv) [Vak14, Exercise 4.5.C (c)] Show that a homogeneous ideal is prime if and only if $ab \in I$ implies $a \in I$ or $b \in I$ for *homogeneous* elements of S.

Definition A.9 (The homogeneous spectrum)**.** Let S be a graded ring. Define $\operatorname{Proj} S$ to be the set of homogeneous prime ideals in S *that do not contain the irrelevant ideal.* This is called the *homogeneous spectrum* of S. For any homogeneous ideal $J \subset S$ we define $V_+(J) \subset \operatorname{Proj} S$ to be the set of homogeneous primes of S containing J and not containing the irrelevant ideal. We define $D_+(J)$ to be the complement of $V_+(J)$ in $\operatorname{Proj} S$.

Exercise A.10 (The universal property of an open subset of the homogeneous spectrum)**.** (i) Suppose $\varphi : S \to T$ is a homomorphism of graded rings such that $\sqrt{\varphi(S_+)T} = T_+$. Show that φ induces a continuous function $\operatorname{Proj} T \to \operatorname{Proj} S$ sending a homogeneous ideal $\mathfrak{p}$ of T to $\varphi^{-1}\mathfrak{p}$.

Correction! Thanks to Jon Lamar for noticing two errors here: the condition $\varphi^{-1}T_+ = S_+$ in the first part and the condition $\sqrt{\varphi(J)T} = T_+$ in the second part were incorrect.

(ii) Suppose that $u : \operatorname{Proj} T \to \operatorname{Proj} S$ is induced from a graded homomorphism $\varphi : S \to T$ as in the first part. Show that u factors through $D_+(J)$ if and only if $\sqrt{\varphi(J)T} \supset T_+$.

Exercise A.11. Suppose that $S_+ = S$. Show that $\operatorname{Proj} S = \operatorname{Spec} S_0$ as a topological space. (Hint: If this is difficult, use the additional assumption that S contains an invertible element of non-zero degree; this is the only case that we will use. It is possible to reduce the general case to this one. The special case is essentially [Vak14, Exercise 4.5.E] or [Sta15, Tag 00JO].)

We give $\operatorname{Proj} S$ the sheaf of rings defined on the basis of open sets of the form $D_+(f)$ *where f has nonzero degree* by

Correction: We only make the definition for f of nonzero degree to make the following exercise easier.

$$\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = S[f^{-1}]_0.$$

Exercise A.12. (i) Show that the open sets $D_+(f)$, for $f \in S_+$, form a basis for the topology of $\operatorname{Proj} S$.

(ii) Verify that $\mathcal{O}_{\operatorname{Proj} S}$ is a presheaf.

(iii) Verify that $\mathcal{O}_{\operatorname{Proj} S}$ is a sheaf on the basis $\mathscr{U} = \{D_+(f)\}$. (Hint: Save yourself work and reduce to Exercise 5.11.)

(iv) Extend $\mathcal{O}_{\operatorname{Proj} S}$ to a sheaf on $\operatorname{Proj} S$ in the only possible way. Show that $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme.

7 Absolute properties of schemes

Most useful properties of schemes are *relative*, meaning they may be applied to families of schemes. We can't talk about relative properties yet since we haven't yet defined morphisms of schemes, so we'll only introduce a limited array of definitions.

Reading 7.1. [Vak14, §§3.3, 3.6], [Har77, §II.3]

7.1 Connectedness

Definition 7.2. A scheme X is *connected* if its underlying topological space is connected.

Worth being aware of, not necessarily important to write up.

Exercise 7.3.

(i) Show that a scheme X is disconnected if and only if $\Gamma(X, \mathcal{O}_X)$ contains an idempotent element other than 0 and 1. (Hint: Let e be an idempotent. Then $D(e)$ and $D(1-e)$ are disjoint open subsets whose union is X.)

(ii) Show that $\operatorname{Spec}(A \times B) = \operatorname{Spec}(A) \amalg \operatorname{Spec}(B)$.

An exercise for the logically oriented. The first 4 parts should be easy. The last is pretty hard.

Exercise 7.4. Let A_i be a collection of commutative rings, indexed by i in a set I.

(i) Construct a map $\coprod \operatorname{Spec} A_i \to \operatorname{Spec} \prod A_i$.

(ii) Show that this map is an isomorphism if I is a finite set. (Hint: Use an earlier exercise.)

(iii) Show that this map is always injective.

(iv) Show that this is *not* an isomorphism if I is infinite. (Hint: Affine schemes are quasicompact.)

(v) Construct an element of $\operatorname{Spec} \prod A_i$ that is not in the image of $\coprod \operatorname{Spec} A_i$. (Hint: I don't suggest attempting this problem unless you know what an ultrafilter is.)

7.2 Quasicompactness

Definition 7.5. A scheme X is said to be *quasicompact* if every open cover of X has a finite subcover.

This is a repeat of Exercise 3.14.

Exercise 7.6. Show that every affine scheme is quasicompact.

Not very important, should be easy.

Exercise 7.7.

1. Construct an example of a scheme that is not quasicompact.

2. Construct an example of a connected scheme that is not quasicompact.

7.3 Quasiseparatedness

Definition 7.8. A scheme X is said to be *quasiseparated* if the intersection of any two quasicompact open subsets of X is quasicompact.

Exercise 7.9. Give an example of a scheme that is not quasiseparated.

7.4 Nilpotents

Definition 7.10. Recall that a commutative ring is said to be *reduced* if it contains no nonzero nilpotent elements. A scheme X is said to be *reduced* if $\mathcal{O}_X(U)$ is a reduced ring for all open subsets U of X.

Exercise 7.11. Let X be any scheme. Construct a reduced scheme X_{red} with the same underlying topological space as X by replacing $\mathcal{O}_X(U)$ by its associated reduced ring.

Definition 7.12. A scheme is said to be *integral* if it is reduced and irreducible.

7.5 Irreducibility

Definition 7.13. A scheme X is *reducible* if its underlying topological space is the union of two closed subsets, neither of which is equal to X. Otherwise it is *irreducible*.

Important, but should be easy.

Exercise 7.14. Show a scheme X is irreducible if and only if every pair of open subsets of X have non-empty intersection.

Important, but should be easy.

Exercise 7.15. Show that an affine scheme $X = \operatorname{Spec} A$ is reducible if and only if A contains a non-nilpotent divisor of zero.

This is a good one to do. It's not extremely important, but it requires putting together a few different ideas without being too difficult.

Exercise 7.16 ([Har77, Proposition II.3.1])**.** Prove that a reduced scheme X is irreducible if and only if $\mathcal{O}_X(U)$ is an integral domain for all open $U \subset X$.

7.6 Noetherian and locally noetherian schemes

Definition 7.17. A scheme that has an open cover by spectra of noetherian rings is called *locally noetherian*. If the cover can be chosen to be finite then the scheme is said to be *noetherian*.

A useful fact to know. The first part should be easy. The second part uses a trick you should become familiar with if you aren't already.

Exercise 7.18.

(i) Show noetherian is equivalent to the conjunction of locally noetherian and quasi-compact.

(ii) Show that $\operatorname{Spec} A$ is locally noetherian if and only if A is a noetherian ring. (Hint: Two ideals that are locally the same are the same.)

This exercise and the next are essentially the same. Think about both, but don't write up more than one of them.

Exercise 7.19. If X is a noetherian scheme then every open subset of X is quasi-compact. (In fact, every subset whatsoever is quasicompact, and the proof isn't any harder.)

Exercise 7.20. Show that the underlying topological space of a noetherian scheme is noetherian: *any* increasing union of open subsets stabilizes.

Important fact!

Exercise 7.21 (Irreducible components)**.** Prove that a noetherian scheme is the union of finitely many irreducible closed subsets. Conclude that a noetherian ring has finitely many minimal prime ideals.

Exercise 7.22 (Noetherian induction (cf. [Har77, Exercise 3.16]))**.** Let X be a noetherian scheme and S a collection of closed subsets of X. Assume that whenever $Z \subset X$ is closed and S contains all proper closed subsets of Z, the set Z also appears in S. Prove that X appears in S.

Exercise 7.23. Find equations defining the union of the 3 coordinate axes in $\mathbf{A}^3$.

7.7 Generic points

Important fact. The argument is fairly straightforward topology, hence not too important to write up.

Exercise 7.24. Each irreducible closed subset Z of a scheme X has a unique point that is dense in Z. This is called the *generic point* of Z.

7.8 Specialization and generization

Definition 7.25. Suppose x and y are points of a scheme X. If y lies in the closure of x then we say x *specializes* to y and y *generizes* to x. We often write $x \rightsquigarrow y$ for this.

Important fact, not important to write up.

Exercise 7.26.

(i) Show that closed subsets are stable under specialization and open subsets are stable under generization.

(ii) Give an example of a subset of a scheme that is stable under generization but not open and an example of a subset that is stable under specialization but not closed.

8 Sheaves II

Reading 8.1. [Vak14, §§2.3, 2.6]

8.1 Pushforward

Definition 8.2 (Pushforward of presheaves)**.** Let $f : X \to Y$ be a continuous function. If F is a presheaf on X then f_*F is the presheaf on Y whose value on and open subset $U \subset Y$ is $f_*F(U) = F(f^{-1}U)$.

Exercise 8.3 (Pushforward of sheaves)**.** The pushforward of a sheaf is a sheaf.

Exercise 8.4 (Pushforward to a point)**.** Let F be a sheaf on a topological space X and let $\pi : X \to (\text{point})$ be the projection to a point. Show that $\pi_*F = \Gamma(X, F)$ when sheaves on a point are regarded as sets.

8.2 Sheaf of sections

Definition 8.5. Let $\pi : Y \to X$ be a continuous function. A *section* of π over a map $f : Z \to X$ is a map $s : Z \to Y$ such that $\pi s = f$. In particular, a section over X is a section over the identity map $\mathrm{id} : X \to X$. We write $\Gamma(Z, Y)$ for the set of sections of $\pi : Y \to X$ over $f : Z \to X$ (leaving the names of the maps implicit).

We define a presheaf Y^{sh} on X by $Y^{\mathrm{sh}}(U) = \Gamma(U, Y)$ for all open $U \subset X$.

Exercise 8.6 (The sheaf of sections)**.** Show that Y^{sh}, as defined above, is a sheaf.

8.3 Espace étalé

Definition 8.7. A function $\pi : E \to X$ is called a *local homeomorphism* or *étale* if there is a cover of E by open subsets U such that $\pi\big|_U : U \to X$ is an open embedding.[2]

[2] The word *étale* is also applied to certain morphisms of schemes, but the definition is different.

A morphism of étale spaces $\pi : E \to X$ and $\pi' : E' \to X$ is a continuous map $f : E \to E'$ such that $\pi' f = \pi$. We write ét(X) for the category of all étale spaces over X.[3]

Exercise 8.8. Show that any map between étale spaces over X is a local homeomorphism.

If F is a presheaf over X, construct a diagram $\mathbf{Open}(X)/F$ whose objects are pairs (U, ξ) where $U \in \mathbf{Open}(X)$ and $\xi \in F(U)$. There is exactly one arrow $(U, \xi) \to (V, \eta)$ whenever $U \subset V$ and $\eta\big|_U = \xi$. Define

$$F^{\text{ét}} = \varinjlim_{(U,\xi) \in \mathbf{Open}(X)/F} U.$$

There is a projection $\pi : F^{\text{ét}} \to X$ by the universal property of the direct limit, setting $U \to X$ to be the inclusion on the (U, ξ) entry.

Exercise 8.9 (The espace étalé)**.** Show that $\pi : F^{\text{ét}} \to X$ is a local homeomorphism. (Hint: Show that the map $U \to F^{\text{ét}}$ associated to $\xi \in F(U)$ is an open embedding using Exercise 8.8 and the fact that such a map is a section.)

Exercise 8.10 (Sheaves and étale spaces are equivalent)**.** Show that the constructions $E \mapsto E^{\text{sh}}$ and $F \mapsto F^{\text{ét}}$ are inverse equivalences between ét(X) and $\mathbf{Sh}(X)$ for any topological space X.

8.4 Associated sheaf

Definition 8.11. If F is any presheaf then $(F^{\text{ét}})^{\text{sh}}$ is a sheaf, called the *associated sheaf* of F. We write $F^{\text{sh}} = (F^{\text{ét}})^{\text{sh}}$ for brevity.

Very important. Writing it up can get technical, so it might be more valuable to think it through than to write your proof down carefully.

Exercise 8.12 (Universal property of the associated sheaf)**.** Let F be a presheaf on a topological space X.

(i) Construct a map $F \to F^{\text{sh}}$ and show that it is universal among maps from F to sheaves.

(ii) Prove that for any sheaf G,

$$\mathrm{Hom}_{\mathbf{Psh}(X)}(F, G) \simeq \mathrm{Hom}_{\mathbf{Sh}(X)}(F^{\text{sh}}, G)$$

in a natural way. (This is really a restatement of the first part.)

8.5 Pullback of sheaves

Definition 8.13 (Fiber product)**.** If $f : X' \to X$ and $p : E \to X$ are continuous functions, a fiber product is a universal topological space E' fitting into a commutative diagram[4]

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "p'"'] & E \arrow[d] \\ X' \arrow[r] & X. \end{tikzcd}$$

We often write $E' = f^{-1}E$ and call E' the pullback of E.

[3] This will cause a technical, but not moral or spiritual, conflict of notation when we study étale morphisms of schemes later.

[4] In fact this definition applies in any category.

Exercise 8.14 (Existence of fiber product in topological spaces)**.** Show that a fiber product can be constructed in topological as the set of all pairs $(x, e) \in X' \times E$ such that $f(x) = p(e)$, topologized as a subspace of the product.

Exercise 8.15 (Pullback of local homeomorphisms)**.** In the notation of Definition 8.13, suppose that $p : E \to X$ is a local homeomorphism. Show that $p' : E' \to X'$ is also a local homeomorphism.

Definition 8.16 (Pullback of sheaves)**.** If $f : X \to Y$ is a continuous map and G is a sheaf on Y then define

$$f^{-1}G = f^{-1}(G^{\text{ét}})^{\text{sh}}$$

Exercise 8.17. Let $f : X \to Y$ be a continuous map of topological spaces, let F be a sheaf on X, and let G be a sheaf on Y. Construct a natural bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}G, F) \simeq \mathrm{Hom}_{\mathbf{Sh}(Y)}(G, f_*F).$$

8.6 Stalks

Definition 8.18. If F is a presheaf over X, write $\pi : F^{\text{ét}} \to X$ for the espace étalé of F. The fiber $\pi^{-1}x$ of $F^{\text{ét}}$ over $x \in X$ is called the *stalk* of F at X and is often denoted F_x.

Exercise 8.19. If F is a presheaf on X, construct a natural isomorphism

$$F_x = \varinjlim_{\substack{U \in \mathbf{Open}(X) \\ x \in U}} F(U).$$

(Hint: One proof of this uses the universal property of f^{-1}, proved in Section 8.5.)

Exercise 8.20. Prove that the stalks of the structure sheaf of a scheme are local rings. (Hint: Reduce immediately to the case of an affine scheme.)

Exercise 8.21. Let η be a point of a topological space X and let ξ be a point of X in the closure of η. Fix a set S and let F be the skyscraper sheaf at η associated to S. (If $j : \eta \to X$ is the inclusion then $F = j_*S$.) Compute the stalk of F at ξ. (If $i : \xi \to X$ is the inclusion then you are computing $i^{-1}j_*S$.)

Chapter 4

The category of schemes

9 Morphisms of schemes

Reading 9.1. [Vak14, §§6.1–6.3,7.1,8.1]

9.1 Morphisms of ringed spaces

Definition 9.2 (Morphisms of ringed spaces)**.** A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a continuous function $\varphi : X \to Y$ and a homomorphism of sheaves of rings $\varphi^* : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$.[1]

Exercise 9.3. Show that we could equivalently have specified a morphism of ringed spaces as a continuous function $\varphi : X \to Y$ and a homomorphism of sheaves of rings $\varphi^* : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Exercise 9.4. (i) Show that a homomorphism of commutative rings $\varphi : B \to A$ induces a morphism of *ringed spaces* $f : \operatorname{Spec} A \to \operatorname{Spec} B$.

(ii) Show that $f^{-1}D(g) = D(\varphi(g))$ for any $g \in A$.

Not difficult, but not important either. Could be good practice with sheaves if you are new to sheaves.

Exercise 9.5. Construct a morphism of ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ that is does not come from a morphism of rings $A \to B$.

9.2 Locally ringed spaces

Suppose that $(X, \mathcal{O}_X)$ is a ringed space and $f \in \Gamma(U, \mathcal{O}_X)$ for some open $U \subset X$. Let $D(f)$ be the largest open subset U of X such that $f\big|_U$ has a multiplicative inverse.

Important to know, less important to do.

Exercise 9.6. Use the sheaf conditions to prove that $D(f)$ exists. (One approach: Let $F(U) = \{g \in \mathcal{O}_X(U) \mid gf = 1\}$. Show that F is a sheaf and that $F(U)$ is either empty or a 1-element set for all U. Conclude that there is an open $V \subset X$ such that $F(U) = 1$ if and only if $V \subset U$.)

Definition 9.7 (Locally ringed space [AGV 3, Exercise IV.13.9])**.** A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that if $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_X)$ and $(f_1, \ldots, f_n) = \Gamma(U, \mathcal{O}_X)$ then $D(f_1) \cup \cdots \cup D(f_n) = U$.

[1] Warning: Other authors often use $\varphi^\sharp$ instead of φ^* here.

Exercise 9.8. (i) Prove that any ringed space with an open cover by locally ringed spaces is a locally ringed space.

(ii) Prove that affine schemes are locally ringed spaces.

(iii) Conclude that all schemes are locally ringed spaces.

This is important to know for schemes (Exercise 8.20), much less important to know for locally ringed spaces. For schemes, the verification should be easy.

Exercise 9.9. (i) Prove that $x \in U$ is in $D(f)$ if and only if the germ of f at x is invertible.

This is the usual definition of a locally ringed space, so this is important to know for the sake of communication. It's not an important exercise.

(ii) Prove that a ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if and only if all of the stalks of $\mathcal{O}_X$ are local rings. (Hint: A commutative ring is local if and only if its non-unit elements form an ideal.)

If $f \in \Gamma(X, \mathcal{O}_X)$ then we can regard $\mathcal{O}_X$ as a sheaf of 'functions' on X: Restrict f to the stalk $\mathcal{O}_{X,\xi}$. Let $\mathfrak{m}_\xi$ be the maximal ideal of $\mathcal{O}_{X,\xi}$. Then the residue of f in the *residue field* $k(\xi) = \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi$ is the *value* of f at ξ.

Exercise 9.10. Show that this definition coincides with the more familiar notion of value for $\xi \in \mathbf{C}^n \subset \operatorname{Spec} \mathbf{C}[x_1, \ldots, x_n]$.

Exercise 9.11. Show that when $f(\xi)$ is interpreted as the value of ξ in the residue field of ξ that $D(f)$ is the set of points ξ where $f(\xi) \neq 0$.

9.3 Morphisms of locally ringed spaces

Definition 9.12 (Morphisms of schemes)**.** If $\varphi : X \to Y$ is a morphism of ringed spaces and both X and Y are locally ringed spaces and $\varphi^{-1}(D_U(f)) = D_{\varphi^{-1}U}(\varphi^* f)$ for any open $U \subset Y$ and any $f \in \Gamma(U, \mathcal{O}_Y)$ then we say φ is a *morphism of locally ringed spaces.*

A *morphism of schemes* $\varphi : X \to Y$ is a morphism of the underlying locally ringed spaces.

Exercise 9.13. Suppose that $f : X \to Y$ is a morphism of ringed spaces.

(i) Suppose there is a cover of Y by open subsets U such that $f^{-1}U \to U$ is a morphism of locally ringed spaces. Show that f is a morphism of locally ringed spaces.

(ii) Suppose that there is a cover of X by open subsets U such that $U \to Y$ is a morphism of locally ringed spaces. Show that f is a morphism of locally ringed spaces.

In other words, if X and Y are schemes then a morphism of ringed spaces $f : X \to Y$ is a morphism of schemes if for each point $x \in X$, there is an open affine neighborhood $U = \operatorname{Spec} A$ of X and an open affine neighborhood $V = \operatorname{Spec} B$ of $f(x)$ such that $f(U) \subset V$ and the map $\operatorname{Spec} A \to \operatorname{Spec} B$ is induced from a homomorphism $B \to A$.

Exercise 9.14 (The usual definition of morphisms of locally ringed spaces [AGV 3, Exercise IV.13.9 c)])**.** With notation as in Definition 9.12, show φ is a morphism of locally ringed spaces if and only if for every point x of X, the map $\varphi^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings. (Recall that this means $\varphi^* \mathfrak{m}_{\varphi(x)} \subset \mathfrak{m}_x$.)

9.4 Morphisms to affine schemes

Theorem 9.15. *If* $(X, \mathcal{O}_X)$ *is a locally ringed space then* $\mathrm{Hom}(X, \operatorname{Spec} A) = \mathrm{Hom}(A, \Gamma(X, \mathcal{O}_X))$ *in a natural way.*

Exercise 9.16. If A and B are commutative rings then

$$\mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} B, \operatorname{Spec} A) = \mathrm{Hom}_{\mathbf{ComRng}}(A, B).$$

Exercise 9.17.

(i) Show that for any affine scheme, $\mathrm{Hom}(\operatorname{Spec} A, \mathbf{A}^1) = A$ in a natural way.

(ii) Show that for any scheme, $\mathrm{Hom}(X, \mathbf{A}^1) = \Gamma(X, \mathcal{O}_X)$.

10 Properties of morphisms

10.1 Nilpotents

Exercise 10.1. Let $A = \mathbf{A}^1$ and let $B = \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2)$. Show B has only one point but that there are non-zero functions that take the value 0 at this point.

10.2 Open embeddings

Exercise 10.2 (Open subschemes)**.** If X is a scheme $U \subset X$ is an open subset, define $\mathcal{O}_U$ to be the restriction of $\mathcal{O}_X$ to U. Show that U is a scheme.

Definition 10.3. A morphism of schemes $U \to X$ is said to be an *open embedding* if it can be factored as $U \to V \to X$ where $U \to V$ is an isomorphism and $V \to X$ is the inclusion of an open subscheme.

10.3 Affine morphisms

Definition 10.4 (Affine morphism)**.** A morphism of schemes $f : X \to Y$ is said to be affine if, whenever $U \subset Y$ is an affine open subset, $f^{-1}U \subset X$ is an affine open subset.

Exercise 10.5. Show that any morphism between affine schemes is affine.

10.4 Closed embeddings

Definition 10.6. A morphism of schemes $f : Z \to X$ is called a *closed embedding* if it is affine and for any affine open subset $U \subset X$ with $U = \operatorname{Spec} A$ and $f^{-1}U = \operatorname{Spec} B$, the map $A \to B$ is a surjection.

Exercise 10.7. Show that $f : Z \to X$ is a closed embedding if and only if it is the inclusion of a closed subset and the map $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Z$ is surjective.

10.5 Locally closed embeddings

Definition 10.8. A *locally closed subscheme* is a closed subscheme of an open subscheme. A morphism of schemes $f : Z \to X$ is called a *locally closed embedding* if it can be factored as a closed embedding followed by an open embedding.

It's valuable to know pathologies like this can exist, less important to have actually seen them.

Exercise 10.9.

(i) Let $X = \mathbf{A}^\infty = \operatorname{Spec} \mathbf{C}[x_1, x_2, \ldots]$ and let $U \subset X$ be the complement of the origin $(0, 0, \ldots)$. (In other words $U = D(x_1, x_2, \ldots)$.) Show that there is a closed subscheme $Y \subset U$ such that $Y \cap D(x_m)$ is

$$Y \cap D(x_m) = \operatorname{Spec} k[x_1, x_2, \ldots, x_m^{-1}]/(x_1^m, x_2^m, \ldots, x_{m-1}^m, x_{m+1}, x_{m+2}, \ldots].$$

(ii) Show that the smallest closed subscheme of X containing Y is X itself.

(iii) Conclude that Y is not an open subscheme of a closed subscheme of X.

This is a surprisingly important fact.

Exercise 10.10. Show that if X is a noetherian scheme then every locally closed subscheme of X is an open subscheme of a closed subscheme of X.

10.6 Relative schemes

We frequently want to distinguish between coefficients in a ring and variables. For example, if you define the coordinate ring of a curve over $\mathbf{C}$ as $\mathbf{C}[x, y]/(f(x, y))$, don't want to think of automorphisms of $\mathbf{C}$ as automorphisms of the curve. We accomplish this algebraically by introducing the category of $\mathbf{C}$-*algebras*.

Definition 10.11. Let A be a commutative ring. An A-*algebra* is a pair (B, φ) where B is a commutative ring and $\varphi : A \to B$ is a homomorphism of commutative rings. A homomorphism of A-algebras $(B, \varphi) \to (C, \psi)$ is a homomorphism of commutative rings $f : B \to C$ such that $f \circ \varphi = \psi$.

In other words, homomorphisms of A-algebras are homomorphisms of commutative rings that hold the coefficient ring constant. If we translate this geometrically, we obtain the notion of a relative scheme:

Definition 10.12. Let A be a commutative ring. An A-*scheme* is a pair (X, π) where $\pi : X \to \operatorname{Spec} A$ is a morphism of schemes. A morphism of A-schemes $(X, \pi) \to (Y, \tau)$ is a morphism of schemes $f : X \to Y$ such that $\tau \circ f = \pi$.

Should be easy. Do it if it's not obvious.

Exercise 10.13. Verify that A-algebras and affine A-schemes are contravariantly equivalent categories. (You'll have to define an affine A-scheme. There are two obvious definitions, both equivalent.)

We can generalize this definition and think about schemes that are constructed using coefficients coming from the structure sheaf of another scheme:

Definition 10.14. Let S be a scheme. An S-*scheme* is a pair (X, π) where $\pi : X \to S$ is a morphism of schemes. A morphism of S-schemes $(X, \pi) \to (Y, \tau)$ is a morphism of schemes $f : X \to Y$ such that $\tau \circ f = \pi$.

Usually when we are working with S-schemes we refer to an S-scheme (X, π) as X and sometimes don't even introduce a letter for π. This shouldn't be a source of confusion, since π is usually clear from context.

10.7 Examples

Exercise 10.15. Show that the disjoint union of two schemes is a scheme in a natural way (you will have to specify the structure sheaf yourself). Show that your construction has the universal property of a coproduct: $\mathrm{Hom}(X \amalg Y, Z) = \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)$ for all Z.

Recommended exercise! This is a better organized version of a problem discussed in class.

Exercise 10.16. Let X be a scheme (in fact, X can be a locally ringed space). Construct a bijection between maps $X \to \mathbf{P}^1$ and the set of quadruples (V_0, V_1, x_0, x_1) such that

(i) $V_i \subset X$ are open subsets of X with $V_0 \cup V_1 = X$,

(ii) $x_i \in \Gamma(V_i, \mathcal{O}_X)$, and

(iii) $x_0\big|_{V_0 \cap V_1} x_1\big|_{V_0 \cap V_1} = 1$.

11 The functor of points

Reading 11.1. [Mum99, §II.6], [Vak14, §§6.6.1–6.6.2, 9.1.6–9.1.7]

11.1 The problem with the product

The world would be unjust if we could not say that

$$\mathbf{A}^1 \times \mathbf{A}^1 = \mathbf{A}^2.$$

Exercise 11.2. (i) Show that

$$|\mathbf{A}^1| \times |\mathbf{A}^1| \neq |\mathbf{A}^2|,$$

where $|X|$ denotes the underlying topological space of a scheme X. (Hint: Find a point of $\mathbf{A}^2$ that does not correspond to an ordered pair of points. Feel free to work over a field, or even an algebraically closed field, where the important phenomenon will already be visible.)

(ii) Show that $\mathbf{A}^2$ has the correct universal property of a product in the category of schemes.[2] (Hint: A map from a scheme X to $\operatorname{Spec} A$ is a homomorphism of commutative rings $A \to \Gamma(X, \mathcal{O}_X)$. Use the universal property of a polynomial ring or a tensor product.)

This tells us that the universal property is a better way of identify products than by looking at the underlying set of points.

11.2 About underlying sets

Reading 11.3. [Mum99, pp. 112–113]

Many mathematical objects of interest have underlying sets. Algebraic objects like rings and groups are defined by adding an algebraic structure to an underlying set. A topological space is an underlying set and a collection of subsets of that set. A manifold is a topological space with additional structure, and its underlying set is the underlying set of the underlying topological space.

[2] This means that a map $X \to \mathbf{A}^2$ corresponds to a pair of maps $X \to \mathbf{A}^1$.

In each of these examples, passage to the underlying set defines a functor, sometimes called a *forgetful functor*:

$$F : \mathscr{C} \to \mathbf{Sets}$$

In fact, all of these functors are *representable*. This means that there is some object $X \in \mathscr{C}$ such that $F \simeq h^X$, where $h^X(Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Important general knowledge, but not particularly important for this class.

Exercise 11.4. Find objects representing the forgetful functors for groups, rings, topological spaces. (Hint: Free object with one generator.)

All of the forgetful functors described above are *faithful*. This means that the map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{Sets}}(FX, FY) \tag{$*$}$$

is injective. In other words, you can tell if two morphisms in $\mathscr{C}$ are the same by looking at how they behave on the underlying sets.

Not every category $\mathscr{C}$ has an 'underlying set' functor to the category of sets, but (essentially) every category does have a faithful functor to the category of sets:

Think about this, but don't write it up. This exercise will be generalized by the Yoneda lemma later.

Exercise 11.5. Let $\mathscr{C}$ be a category. Prove that the functor

$$F(Y) = \prod_{X \in \mathscr{C}} \mathrm{Hom}(Y, X)$$

is a faithful functor from $\mathscr{C}$ to **Sets**. (If you are the kind of person who likes to worry about set-theoretic issues, assume that $\mathscr{C}$ is a small category so that the product exists.)

Of course, the forgetful functors we have encountered are not full. To be *full* means that the map ($*$) is a surjection. However, the functors above can be promoted to fully faithful functors by recording extra structure:

This exercise won't serve any further purpose in this course, so you shouldn't do it. It is important in the construction of cohomology theories for algebraic structures, though.

Exercise 11.6. In the following problems, you will need to figure out what 'compatible' means.

(i) Show that compatible functions

$$\mathrm{Hom}_{\mathbf{ComRing}}(\mathbf{Z}[x], A) \to \mathrm{Hom}_{\mathbf{ComRing}}(\mathbf{Z}[x], B)$$
$$\mathrm{Hom}_{\mathbf{ComRing}}(\mathbf{Z}[x, y], A) \to \mathrm{Hom}_{\mathbf{ComRing}}(\mathbf{Z}[x, y], B)$$

are induced by a unique homomorphism of commutative rings $A \to B$. (Hint: Use the map $\mathbf{Z}[x] \to \mathbf{Z}[x, y]$ sending x to $x + y$ and the map sending x to xy.)

(ii) Let F_n denote the free group on n generators. Show that compatible functions

$$\mathrm{Hom}_{\mathbf{Grp}}(F_1, A) \to \mathrm{Hom}_{\mathbf{Grp}}(F_1, B)$$
$$\mathrm{Hom}_{\mathbf{Grp}}(F_2, A) \to \mathrm{Hom}_{\mathbf{Grp}}(F_2, B)$$

are induced by a unique homomorphism of groups $A \to B$.

It is harder to come up with a fully faithful embedding of topological spaces into a category that is similarly set-theoretic, and harder still to do it for schemes. However, in the next section, we will see that *every* category has a fully faithful functor to a category that is essentially set-theoretic (meaning its objects are sets with structural morphisms between them). In other words, morphisms in any category can be constructed set-theoretically. This can be quite a coup for categories like schemes, where the definition of the category is very elaborate.

11.3 Yoneda's lemma

Definition 11.7. If C is a category, a *presheaf* on C is a contravariant functor from C to **Sets**. If $X \in C$ then we write h_X for the functor $h_X(Y) = \mathrm{Hom}_C(Y, X)$. If F is a presheaf on C and $F \simeq h_X$ then we say F is *representable* by X.

Exercise 11.8 (Yoneda's Lemma)**.** (i) Let C be a category. Show that $X \mapsto h_X$ is a covariant functor from C to $\widehat{C}$.

(ii) Show that $X \mapsto h_X$ is fully faithful. (Show in other words that $\mathrm{Hom}_C(X, Y) = \mathrm{Hom}_{\widehat{C}}(h_X, h_Y)$ via the natural map.)

(iii) Show that for any $F \in \widehat{C}$, there is a unique natural bijection $\mathrm{Hom}_{\widehat{C}}(h_X, F) \simeq F(X)$ under which $\varphi : h_X \to F$ corresponds to $\varphi(\mathrm{id}_X) \in F(X)$.

Yoneda's lemma tells us that we can think of a scheme in terms of the contravariant functor it represents. Remarkably this can often be a lot easier than thinking about the scheme as a ringed space.

Chapter 5

Representable functors

12 Presheaves representable by schemes

Reading 12.1. [Vak14, §§9.1.6–9.1.7]

Recall that the Yoneda lemma gave us a fully faithful functor

$$\mathbf{Sch} \to \mathbf{Sch}^{\wedge}$$

where $\mathbf{Sch}^{\wedge}$ is the category of presheaves on $\mathbf{Sch}$. In this section, we want to characterize the image of this functor. In other words, we want to be able to determine which presheaves on $\mathbf{Sch}$ are representable by schemes. This will give us a new way to construct schemes that will often be easier than constructing a ringed space. In fact, this will give us an entirely new way to think about what a scheme is.

Theorem 12.2. *A presheaf on the category of schemes is representable by a scheme if and only if*

(i) it is a sheaf in the Zariski topology, and

(ii) it has an open cover by presheaves that are representable by affine schemes.

If A is a commutative ring then $h^A : \mathbf{ComRing} \to \mathbf{Sets}$ sending B to $\mathrm{Hom}_{\mathbf{ComRing}}(A, B)$. If X is a scheme then $h_X : \mathbf{Sch}^{\circ} \to \mathbf{Sets}$ is the functor sending Y to $\mathrm{Hom}_{\mathbf{Sch}}(Y, X)$. Abusively, we think of h^A and $h_{\mathrm{Spec}\, A}$ as being the same object. In reality, h^A is a functor defined on $\mathbf{ComRing} = \mathbf{Aff}^{\circ} \subsetneq \mathbf{Sch}^{\circ}$ and $h_{\mathrm{Spec}\, A}$ is defined on all of $\mathbf{Sch}$. More generally, when X is a scheme, we sometimes think of h_X as a covariant functor defined on $\mathbf{ComRing}$ and we abbreviate $h_X(\mathrm{Spec}\, A)$ to $h_X(A)$. We will see below that the composition of the Yoneda embedding with restriction

$$\mathbf{Sch} \to \mathbf{Sch}^{\wedge} \to \mathbf{Aff}^{\wedge}$$

is fully faithful, so this abuse of notation does not cause any trouble. Later on, we will even permit ourselves to write $X(A)$ in place of $h_X(A)$.

12.1 Zariski sheaves

Let Y be a scheme. Note that $\mathbf{Open}(Y)$ can be regarded as a subcategory of $\mathbf{Sch}$. If F is a presheaf on $\mathbf{Sch}$ then we can restrict it to $\mathbf{Open}(Y)$ and get a presheaf on Y.

Definition 12.3. A presheaf F on $\mathbf{Sch}$ is said to be a *Zariski sheaf* if, for any scheme Y, the presheaf $F\big|_{\mathbf{Open}(Y)}$ is a sheaf on Y.

The following lemma says that h_X is a Zariski sheaf for any scheme X:

Lemma 12.4. *Suppose X and Y are schemes. Define a presheaf F on X by $F(U) = \mathrm{Hom}(U, Y)$. Then F is a sheaf. (Hint: Use Exercise 4.9 and Exercise 4.12. It may be helpful to think of a map of ringed spaces as a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Observe that if $U \subset X$ is open then $f^{-1}\mathcal{O}_Y\big|_U = (f\big|_U)^{-1}\mathcal{O}_Y$.)*

Exercise 12.5. Suppose that F is a presheaf on $\mathbf{Sch}$. Show that there is a universal map $F \to F^{\mathrm{sh}}$ where F^{sh} is a Zariski sheaf. This is called the *sheafification* of F. (Hint: Sheafify $F\big|_{\mathbf{Open}(X)}$ for each $X \in \mathbf{Sch}$.)

12.2 Open subfunctors

Should be simple

Exercise 12.6. If $\varphi : F \to G$ is a natural transformation between presheaves and $G' \subset G$ is a subpresheaf then define $F'(U) = \varphi^{-1}G'(U)$ for all U. Show that F' is a subpresheaf of F. We denote $F' = \varphi^{-1}G'$.

This exercise is a special case of Exercise 12.5, although we'll only have use for this one right now. The first part is not so important to do, so much as to know. The second part might be good practice.

Exercise 12.7. Suppose that F is a sheaf and G_i, $i \in I$ is a family of subsheaves of F. Let $G(X) = \bigcup_{i \in I} G_i(X)$. Say that $\xi \in F(X)$ *lies locally* in G if there is an open cover $X = \bigcup U_j$ such that for each j, the restriction $\xi\big|_{U_j}$ lies in $G(U_j)$.

(i) Show that G is not necessarily a sheaf.

(ii) Show that there is a smallest subsheaf G' of F that contains all of the G_i. (Hint: Let $G'(X)$ be the set of all $\xi \in F(X)$ that lie locally in G.)

(iii) Suppose that $F = h_X$ is representable. Show that $G' = F$ if and only if id_X lies locally in G. (Hint: h_X is the only subpresheaf of itself containing id_X.)

The sheaf constructed in part (ii) is called the *sheaf theoretic union* or the *sheaf union* of the G_i.

Definition 12.8. Suppose that F is a presheaf on the category of schemes and $F' \subset F$ is a subpresheaf. We say that F' is *open* in F if, for any map $\varphi : h_X \to F$, the preimage $\varphi^{-1}F' \subset h_X$ is representable by an open subscheme of X.

A collection of open subfunctors $F'_i \subset F$ is said to *cover* F if F is the sheaf theoretic union of the F'_i.

This might be a good exercise to do. It will force you to unpack some of the definitions. You don't necessarily have to prove all of the equivalences.

Exercise 12.9. Let F be a presheaf on $\mathbf{Sch}$ and $F'_i \subset F$ open subpresheaves. Prove that the following conditions are equivalent:

(i) F is the sheaf union of the F'_i.

(ii) h_X is the sheaf union of the $\varphi^{-1}F'_i$ for all $\varphi \in F(X) = \mathrm{Hom}(h_X, F)$.

(iii) For any scheme X and any $\varphi \in F(X) = \mathrm{Hom}(h_X, F)$ let $U_i \subset X$ be open subschemes such that $\varphi^{-1}F'_i = h_{U_i}$. Then $X = \bigcup U_i$.

(iv) $F(k) = \bigcup_i F_i'(k)$ for all fields k.

Reality check. Shouldn't be difficult.

Exercise 12.10. Show that every scheme has an open cover by subfunctors that are representable by affine schemes.

Lemma 12.11 ([Vak14, Exercise 9.1.I])**.** *If F is a Zariski sheaf on schemes that has an open cover by affine schemes then F is representable by a scheme.*

12.3 The basis of affines

Since every scheme has an open cover by affine schemes, the full subcategory $\mathbf{Aff} \subset \mathbf{Sch}$ behaves a lot like a basis, at least with respect to Zariski sheaves:

Exercise 12.12. Show that a Zariski sheaf on **Aff** extends in a unique way (up to unique isomorphism) to a Zariski sheaf on **Sch**.

Using this we can get another perspective on what a scheme is.

Definition 12.13. Let A be a commutative ring and let $h^A : \mathbf{ComRng} \to \mathbf{Sets}$ be the functor represented by A. For any subset $J \subset A$, let $h_{D(J)}$ be the subfunctor of h^A defined as follows:

$$h_{D(J)}(B) = \{\varphi : A \to B \mid \varphi(J)B = B\}.$$

A subfunctor of h^A is called *open* if it is isomorphic to $h_{D(J)}$ for some subset $J \subset A$.

Warning: $h_{D(J)}$ usually is not representable by a commutative ring!

Exercise 12.14. Show that if h^A is regarded as a contravariant functor $\mathbf{Aff} \to \mathbf{Sets}$ then $h_{D(J)}$ is represented by the subscheme $D(J) \subset \operatorname{Spec} A$, whence the notation.

Not really important

Exercise 12.15.

(i) Show that the intersection of two open subfunctors is an open subfunctor.

(ii) Show that the union of two open subfunctors is not necessarily an open subfunctor.

Exercise 12.16. Suppose that $F \subset G$ is an inclusion of Zariski sheaves and F has an open cover by subfunctors that are also open subfunctors of G. Prove that F is an open subfunctor of G.

Definition 12.17. A morphism $F \to G$ of presheaves on **Aff** is said to be an *open embedding* if, for every morphism $\varphi : h^A \to G$, the preimage $\varphi^{-1}F \subset h^A$ is an open subfunctor.

Definition 12.18. A morphism $F \to G$ of presheaves is said to be a *cover* (with respect to schematic points) if $F(k) \to G(k)$ is a bijection for all fields k.

Definition 12.19 (Alternate definition of a scheme)**.** A presheaf F on **Aff** is called a *scheme* if it is a Zariski sheaf and has a cover by open, representable subfunctors.

Exercise 12.20. Show that the two definitions of schemes (via ringed spaces and via presheaves) yield equivalent categories.

12.4 Fiber products

Reading 12.21. [Vak14, §§9.1–9.3]

Suppose that $p : X \to Z$ and $q : Y \to Z$ are morphisms of schemes. Define $F = h_X \times_{h_Z} h_Y$. That is,

$$F(W) = \{(f, g) \in \operatorname{Hom}(W, X) \times \operatorname{Hom}(W, Y) \bigm| pf = qg \in \operatorname{Hom}(W, Z)\}.$$

Exercise 12.22. Prove that F is a Zariski sheaf.

Exercise 12.23. If $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} C$, and $Z = \operatorname{Spec} A$ then $F \simeq h_{\operatorname{Spec}(B \otimes_A C)}$.

Exercise 12.24. Show that F has an open cover by functors representable by affine schemes. (Hint: For any point $\xi \in F(k)$, choose open affine neighborhoods $U \subset X$, $V \subset Y$, and $W \subset Z$ containing the images of ξ, with $p(U) \subset W$ and $q(V) \subset W$. Let $f : F \to h_X$ and $g : F \to h_Y$ denote the projections. Show that $f^{-1}h_U \cap g^{-1}h_V$ is open in F and affine.)

Fibers

Suppose that $p : X \to S$ is a morphism of schemes. The fiber of p over a point $\xi \in S$ is the fiber product $X \times_S \operatorname{Spec} \mathbf{k}(\xi)$.

Equalizers and the diagonal

Important! **Exercise 12.25.** Suppose that $f, g : X \to Y$ are two morphisms of schemes. Show that there is a universal map $h : Z \to X$ such that $fh = gh$. This is called the *equalizer* of f and g and is sometimes denoted $\operatorname{eq}(f, g)$. (Hint: One way to do this is to construct a sheaf and find an open cover by representable functors. Another way is to build the equalizer using fiber products. It's valuable to think about it both ways, but the second is more common in the algebraic geometry literature.)

12.5 Examples

Exercise 12.26. For any scheme X, let $\mathbf{G}_m(X) = \Gamma(X, \mathcal{O}_X)^*$.

(i) Show directly that $\mathbf{G}_m$ is a Zariski sheaf.

(ii) Show that $\mathbf{G}_m$ is in fact representable by an affine scheme.

13 Vector bundles

Contrary to what we've come to expect, the definition of a vector bundle in algebraic geometry in exactly the same way as in differential geometry or topology. We will give this definition, as well as two others, one aligned philosophically with thinking of schemes as locally ringed spaces, and the other aligned with thinking in terms of the functor of points.

13.1 Transition functions

In differential geometry, a vector bundle over a manifold S is usually defined as a projection $p : E \to S$ along with

(i) a cover $\mathscr{U}$ of S along with *specified isomorphisms* $p^{-1}U \simeq U \times V$ over U for each $U \in \mathscr{U}$, where V is a vector space, possibly depending on U, such that

(ii) if U_1 and U_2 are two open sets in $\mathscr{U}$, the transition function

$$(U_1 \cap U_2) \times V_1 \xrightarrow{\sim} p^{-1}(U_1 \cap U_2) \xrightarrow{\sim} (U_1 \cap U_2) \times V_2$$

is a family of linear maps.[1]

This definition makes sense when S is a scheme. We just need to say explicitly what we mean by a 'vector space' and a 'family of linear maps'. By a vector space, we will simply mean $\mathbf{A}^r$ or a scheme isomorphic to it. A family of linear maps $U \times \mathbf{A}^r \to U \times \mathbf{A}^s$ is a morphism that is given in coordinates on each $\operatorname{Spec} A$ in an affine open cover of U in the form

$$\begin{gathered} A[t_1, \ldots, t_s] \to A[t_1, \ldots, t_r] \\ t_i \mapsto t_i M \end{gathered}$$

for some $M \in \mathrm{Mat}_{s\times r}(A)$.

Definition 13.1 (Vector bundle, version 1)**.** Let S be a scheme. A *vector bundle* over S is a projection $p : E \to S$, along with

(i) a cover $\mathscr{U}$ of S by open subschemes and, for each $U \in \mathscr{U}$, an isomorphism $p^{-1}E \simeq U \times V$ over U for each $U \in \mathscr{U}$, where V is some affine space $\mathbf{A}^r$, such that

(ii) if U_1 and U_2 are two open sets in $\mathscr{U}$, the transition function

$$(U_1 \cap U_2) \times V_1 \xrightarrow{\sim} p^{-1}(U_1 \cap U_2) \xrightarrow{\sim} (U_1 \cap U_2) \times V_2$$

is a family of linear maps over $U_1 \cap U_2$.

13.2 Locally free sheaves

Reading 13.2. [Vak14, Section 13.1]

Definition 13.3 (Locally free sheaf)**.** Let S be a scheme. A sheaf of $\mathcal{O}_S$-modules is a sheaf $\mathscr{E}$, along with the structure of a $\mathcal{O}_S(U)$-module on $\mathscr{E}(U)$ for each open $U \subset S$, such that the restriction maps are equivariant in the sense illustrated below:

$$\begin{tikzcd} \mathcal{O}_S(U) \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & \mathcal{O}_S(V) \arrow[d, phantom, "\circlearrowleft"] \\ \mathscr{E}(U) \arrow[r] & \mathscr{E}(V) \end{tikzcd}$$

A *locally free sheaf* over S is a sheaf of $\mathcal{O}_S$-modules $\mathscr{E}$ such that $\mathscr{E}$ is locally isomorphic to $\mathcal{O}_S^{\oplus n}$ for some n. If $\mathscr{E}$ is locally isomorphic to $\mathcal{O}_S^{\oplus n}$ then $\mathscr{E}$ is said to be locally free *of rank* n. Locally free sheaves of rank 1 are also called *invertible sheaves.*

[1] This means that the map is of the form $(x, y) \mapsto (x, F(x)y)$ where $F : U_1 \cap U_2 \to \mathrm{Hom}_{\mathbf{Vect}}(V_1, V_2)$ is a C^∞ function. In other words, a family of linear maps from $\mathbf{R}^n$ to $\mathbf{R}^m$ over U is a $m \times n$ matrix of C^∞ functions on U.

In other words, $\mathscr{E}$ is locally free if there is a cover of S by open subsets U such that $\mathscr{E}\big|_U \simeq \mathcal{O}_U^{\oplus n}$ as a sheaf of $\mathcal{O}_U$-modules. Note that the number n does not have to be the same for every open subset in the cover.

13.3 Vector space schemes

Exercise 13.4. Show that, for any scheme X, the set $\mathrm{Hom}_{\mathbf{Sch}}(X, \mathbf{A}^1)$ has the structure of a commutative ring, and for any morphism of schemes $X \to Y$, the induced map

$$\mathrm{Hom}_{\mathbf{Sch}}(Y, \mathbf{A}^1) \to \mathrm{Hom}_{\mathbf{Sch}}(X, \mathbf{A}^1)$$

is a ring homomorphism. Interpret this by saying $\mathbf{A}^1$ is a *commutative ring scheme*. (Hint: $\mathrm{Hom}(X, \mathbf{A}^1) = \Gamma(X, \mathcal{O}_X)$.)

Definition 13.5. An *scheme of* $\mathbf{A}^1$*-modules*[2] *over a scheme* S is an S-scheme E and the structure of a $\mathrm{Hom}_{\mathbf{Sch}/S}(T, \mathbf{A}^1)$-module on $\mathrm{Hom}_{\mathbf{Sch}/S}(T, E)$ for every S-scheme T, such that for every morphism of S-schemes $f : U \to T$, the function

$$\mathrm{Hom}_{\mathbf{Sch}/S}(T, E) \to \mathrm{Hom}_{\mathbf{Sch}/S}(U, E)$$

is a homomorphism, in the sense that for all $x \in \mathrm{Hom}_{\mathbf{Sch}/S}(T, E)$ and all $\lambda \in \mathrm{Hom}(T, \mathbf{A}^1)$, we have

$$f^*(\lambda x) = f^*(\lambda) f^*(x).$$

A *morphism of schemes of* $\mathbf{A}^1$*-modules over* S is a morphism of S-schemes $E \to F$ such that for any S-scheme T, the map

$$E(T) \to F(T)$$

is an $\mathbf{A}^1(T)$-module homomorphism.

This exercise is a special case of the next one. Do you see how?

Exercise 13.6. Show that there is a natural structure of a scheme of $\mathbf{A}^1$-modules over S on $S \times \mathbf{A}^n$ for any n. (Hint: Show that $\mathrm{Hom}_{\mathbf{Sch}/S}(T, S \times \mathbf{A}^n) = \Gamma(T, \mathcal{O}_T^n)$.) We write $\mathbf{A}_S^n$ for this scheme of $\mathbf{A}^1$-modules.

Exercise 13.7. Suppose that E is a $\mathbf{A}^1$-module over S and $T \to S$ is a morphism of schemes. Put a $\mathbf{A}^1$-module structure on $T \times_S E$ in a natural way.

When $T \subset S$ is an open subscheme, we write $E\big|_T$ for the construction from the previous exercise.

Definition 13.8 (Vector bundle, version 2)**.** A *vector bundle* is a $\mathbf{A}^1$-module E over a scheme S such that there is an open cover of S by schemes T with $E\big|_T \simeq T \times \mathbf{A}^n$ for some n. A *morphism of vector bundles* is a morphism of schemes of $\mathbf{A}^1$-modules.

14 Quasicoherent sheaves and schemes in modules

14.1 Quasicoherent sheaves

This is important but should feel like repetition of Exercise 5.11. Reuse as much of that exercise as you can.

Exercise 14.1. Let $S = \operatorname{Spec} A$ be an affine scheme. If M is an A-module, define $\widetilde{M}(D(f)) = M_f$ where M_f denotes the $A[f^{-1}]$-module $A[f^{-1}] \otimes_A M$.

[2] There doesn't seem to be standard terminology for this object.

(i) Construct restriction morphisms making $\widetilde{M}$ into a presheaf of $\mathcal{O}_S$-modules on the basis of principal open affine subsets of S.

(ii) Show that $\widetilde{M}$ is a sheaf on the basis of principal open affine subsets of S. (Hint: The proof is exactly the same as the proof in Exercise 5.11.)

(iii) Extend $\widetilde{M}$ to a sheaf on $\operatorname{Spec} A$.

Definition 14.2 (Quasicoherent sheaf)**.** A sheaf $\mathscr{F}$ of $\mathcal{O}_S$-modules on a scheme S is said to be *quasicoherent* if there is a basis of affines $U = \operatorname{Spec} A$ such that $\mathscr{F}\big|_U \simeq \widetilde{M}$ for some A-module M.

Exercise 14.3. Show that a sheaf of $\mathcal{O}_S$-modules is quasicoherent if and only if it may be presented locally as the cokernel of a homomorphism of free modules (not necessarily of finite rank).

Note the correction! The word quasicoherent was previously missing.

Exercise 14.4. Show that a quasicoherent sheaf of $\mathcal{O}_S$-modules on an affine scheme $\operatorname{Spec} A$ is always of the form $\widetilde{M}$ for some A-module M.

14.2 Morphisms of vector bundles

Charts

If $p : E \to S$ is a vector bundle with charts $p^{-1}U_i \simeq \mathbf{A}^r_{U_i}$ and $q : F \to S$ is a vector bundle with charts *over the same open sets* $q^{-1}U_i \simeq \mathbf{A}^s_{U_i}$ then a morphism of vector bundles $E \to F$ is a morphism of S-schemes such that the induced maps

$$\mathbf{A}^r_{U_i} \simeq p^{-1}U_i \to q^{-1}U_i \simeq \mathbf{A}^s_{U_i}$$

are linear maps.

Not recommended! The point is that this isn't a pleasant thing to do.

Exercise 14.5. Define a morphism of vector bundles E and F whose charts are given on *different* open covers $\{U_i\}$ and $\{V_j\}$.

Locally free sheaves

Definition 14.6. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of $\mathcal{O}_S$-modules then a homomorphism $\mathscr{F} \to \mathscr{G}$ is a homomorphism of sheaves such that for each open $U \subset S$ the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathcal{O}_S(U)$-modules.

Schemes of modules

Definition 14.7. Suppose E and F are schemes of $\mathbf{A}^1$-modules over S. A morphism $E \to F$ is a morphism of S-schemes $\varphi : E \to F$ such that for every S-scheme T, the map $E(T) \to F(T)$ is $\mathbf{A}^1(T)$-linear.

14.3 Pullback of vector bundles and sheaves

Charts

Suppose $p : E \to S$ is a vector bundle with charts $p^{-1}U_i \simeq \mathbf{A}^r_{U_i}$. Let $f : T \to S$ be a morphism of schemes. Then $q : f^{-1}E \to T$ can be given charts $q^{-1}(f^{-1}U_i) \simeq \mathbf{A}^r_{f^{-1}U_i}$.

Exercise 14.8. Verify that the charts for $f^{-1}E$ are compatible and yield a vector bundle on T.

Schemes of modules

Suppose $p : E \to S$ is a scheme of $\mathbf{A}^1$-modules over S and $f : T \to S$ is a morphism of schemes. For any T-scheme $g : U \to T$, define

$$f^{-1}E(U, g) = E(U, fg).$$

Should be a matter of bookkeeping. Probably not worth writing up.

Exercise 14.9. Show that $f^{-1}E$ is naturally equipped with the structure of a sheaf of $\mathbf{A}^1$-modules over T.

Locally free sheaves

Exercise 14.10 (Pushforward of sheaves of modules)**.** Suppose that $f : X \to Y$ is a morphism of sheaves and $\mathscr{F}$ is a $\mathcal{O}_X$-module. Show that $f_*\mathscr{F}$ is naturally equipped with the structure of a $\mathcal{O}_Y$-module. Show that this gives a functor

$$f_* : \mathcal{O}_X\text{-}\mathbf{Mod} \to \mathcal{O}_Y\text{-}\mathbf{Mod}$$

called *pushforward* of $\mathcal{O}_X$-modules to Y.

Definition 14.11 (Pullback of sheaves of modules)**.** Let $f : X \to Y$ be a morphism of schemes. The *pullback* of an $\mathcal{O}_Y$-module $\mathscr{G}$ is an $\mathcal{O}_X$-module $f^*\mathscr{G}$ with the following universal property: for all $\mathcal{O}_X$-modules $\mathscr{F}$,

$$\mathrm{Hom}_{\mathcal{O}_X\text{-}\mathbf{Mod}}(f^*\mathscr{G}, \mathscr{F}) \simeq \mathrm{Hom}_{\mathcal{O}_Y\text{-}\mathbf{Mod}}(\mathscr{G}, f_*\mathscr{F})$$

naturally in $\mathscr{F}$.

The pullback exists for all sheaves of modules and all morphisms of ringed spaces, but we'll just construct it for quasicoherent sheaves and morphisms of schemes.

Exercise 14.12. (i) Suppose that $f : X \to Y$ is a morphism of *affine* schemes. Construct $f^*\mathscr{F}$ for any quasicoherent sheaf on Y. (Hint: Assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $\mathscr{F} = \widetilde{M}$ and take $f^*\mathscr{F} = (B \otimes_A M)^\sim$.)

(ii) Suppose that $f : X \to Y$ is an arbitrary morphism of schemes and $\mathscr{F}$ is a sheaf of modules on Y. Suppose that you know $(f\big|_U)^*\mathscr{F}$ exists for all U in an open cover of X. Glue these together to construct $f^*\mathscr{F}$.

(iii) Conclude that $f^*\mathscr{F}$ exists whenever $f : X \to Y$ is a morphism of schemes and $\mathscr{F}$ is quasicoherent.

If you already know about the tensor product of sheaves of modules, the following definition of f^* is more efficient than the one above:

Definition 14.13. Suppose that $\mathscr{F}$ is a sheaf of $\mathscr{O}_Y$-modules on Y and $f : X \to Y$ is a morphism of ringed spaces. Define $f^*\mathscr{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{F}$.

Exercise 14.14. Show that $f^*\mathscr{F}$ as defined above satisfies the required universal property.

Should just be a line or two.

Exercise 14.15. Suppose that $\mathscr{F}$ is a locally free sheaf on Y and $f : X \to Y$ is a morphism of schemes. Show that $f^*\mathscr{F}$ is locally free.

Interpretation of sheaf pullback in terms of charts

Fix a map $f : X \to Y$ and a *locally free sheaf* $\mathscr{F}$ on Y. Choose an open cover of Y by U_i and isomorphisms $\alpha_i : \mathscr{F}\big|_{U_i} \simeq \mathcal{O}_{U_i}^{r_i}$. We construct a sheaf of $\mathcal{O}_X$-modules on X by gluing. Take $\mathscr{G}_{f^{-1}U_i} = \mathcal{O}_{f^{-1}U_i}^{r_i}$. On $f^{-1}U_i \cap f^{-1}U_j$, choose the isomorphism

$$\mathscr{G}_{f^{-1}U_i}\big|_{f^{-1}U_i \cap f^{-1}U_j} \simeq \mathcal{O}^{r_i}_{f^{-1}U_i \cap f^{-1}U_j} \to \mathcal{O}^{r_j}_{f^{-1}U_i \cap f^{-1}U_j} \simeq \mathscr{G}_{f^{-1}U_j}\big|_{f^{-1}U_i \cap f^{-1}U_j}$$

to be given by $f^*\varphi_{ij}$, where φ_{ij} is the transition function

$$\varphi_{ij} : \mathcal{O}^{r_i}_{U_i \cap U_j} \xleftarrow{\alpha_i} \mathscr{F}\big|_{U_i}\big|_{U_i \cap U_j} = \mathscr{F}\big|_{U_j}\big|_{U_i \cap U_j} \xrightarrow{\alpha_j} \mathcal{O}^{r_j}_{U_i \cap U_j}.$$

Exercise 14.16. Show that the $\mathscr{G}_{f^{-1}U_i}$ glue together to give $f^*\mathscr{F}$ (via a canonical isomorphism).

Thus the transition functions of $f^*\mathscr{F}$ are pulled back from the transition functions of $\mathscr{F}$. This is one reason it is reasonable to use the notation $f^*\mathscr{F}$ for this construction.

Chapter 6

Some moduli problems

15 Basic examples

15.1 The scheme in modules associated to a quasicoherent sheaf

Reading 15.1. [GD71, §9.4]

Let $\mathscr{F}$ be a quasicoherent sheaf of $\mathcal{O}_S$-modules on S.[1] For each S-scheme T, let

$$F(T) = \mathrm{Hom}_{\mathcal{O}_T\text{-}\mathbf{Mod}}(\mathscr{F}\big|_T, \mathcal{O}_T).$$

We give $F(T)$ the structure of a $\mathbf{A}^1(T)$-module. Suppose that $\lambda \in \mathbf{A}^1(T) = \Gamma(T, \mathcal{O}_T)$ and $x \in F(T)$. Then multiplication by λ gives a morphism $\mathcal{O}_T \to \mathcal{O}_T$ and composition with this homomorphism induces a map $F(T) \to F(T)$. We declare that $\lambda.x$ is the image of x under this map.

We write $F = \mathbf{V}(\mathscr{F})$ for this construction.

Exercise 15.2. Show that $F = \mathbf{V}(\mathscr{F})$ has the structure of a scheme of modules over S:

(i) Prove that F is a Zariski sheaf on $\mathbf{Sch}/S$.

(ii) Prove that F is representable by an affine scheme when $\mathscr{F}$ is quasicoherent and S is affine. (Hint: When $S = \operatorname{Spec} A$ and $F = \widetilde{M}$, represent it by $\operatorname{Spec}\operatorname{Sym} M$.)

(iii) Conclude that F is representable by a scheme over S.

Theorem 15.3. *The functor $\mathbf{V} : \mathbf{QCoh}(S)^\circ \to \mathbf{A}^1\text{-}\mathbf{Mod}/S$ constructed above is fully faithful. It induces an equivalence between the categories of locally free $\mathcal{O}_S$-modules and of vector bundles.*

What we need to show is that the natural map

$$\mathrm{Hom}_{\mathcal{O}_S\text{-}\mathbf{Mod}}(\mathscr{E}, \mathscr{F}) \to \mathrm{Hom}_{\mathbf{A}^1\text{-}\mathbf{Mod}/S}(\mathbf{V}(\mathscr{F}), \mathbf{V}(\mathscr{E})) \qquad (*)$$

is a bijection for any two quasicoherent sheaves $\mathscr{E}$ and $\mathscr{F}$ on S.

[1] In fact, the definition works for any sheaf of $\mathcal{O}_S$-modules.

Exercise 15.4. Show that (19.2) may be regarded as the map of global sections between two sheaves on S. Conclude that to prove (19.2) is a bijection it is sufficient to assume S is affine.

The exercise tells us we may assume that $S = \operatorname{Spec} A$. Then $\mathscr{E} = \widetilde{M}$ and $\mathscr{F} = \widetilde{N}$ for two A-modules M and N. The the underlying schemes of $\mathbf{V}(\widetilde{M})$ and $\mathbf{V}(\widetilde{N})$ are $\operatorname{Spec}\operatorname{Sym}_A M$ and $\operatorname{Spec}\operatorname{Sym}_A N$, respectively.

Exercise 15.5. Prove that every $\mathbf{A}^1$-*linear* map $\mathbf{V}(\widetilde{N}) \to \mathbf{V}(\widetilde{M})$ arises from a homomorphism of A-modules $M \to N$.

16 The projective line

For any scheme S, let $P^1(S)$ be the set of triples (L, s, t) where L is a rank 1 vector bundle on S and $s, t : L \to \mathbf{A}^1_S$ are linear maps such that

$$(s, t) : L \to \mathbf{A}^2_S$$

is a closed embedding. Let $Q^1(S)$ be the set of triples $(\mathscr{L}, x, y)$ where $\mathscr{L}$ is an invertible sheaf on S (Definition 13.3) and $x, y \in \Gamma(S, \mathscr{L})$ are generators of $\mathscr{L}$.[2]

Exercise 16.1. Describe restriction maps making P^1 and Q^1 into functors. (Hint: The pullback of a closed embedding is a closed embedding.)

Exercise 16.2. Prove that P^1 and Q^1 are isomorphic functors.

Exercise 16.3. Prove that P^1 and Q^1 are Zariski sheaves. (Hint: In view of Exercise 16.2, you only have to show one is a Zariski sheaf.)

Exercise 16.4. Let $U \subset Q^1$ be the subfunctor consisting of all triples $(\mathscr{L}, x, y)$ such that x generates $\mathscr{L}$.

(i) Show that the corresponding subfunctor $V \subset P^1$ consists of all triples (L, s, t) such that $s : L \to \mathbf{A}^1$ is an isomorphism.

(ii) Prove that $U \simeq \mathbf{A}^1$. (Hint: On U, multiplication by x gives an isomorphism $\mathcal{O} \to \mathscr{L}$.)

(iii) Show that U is an open subfunctor of Q^1.

(iv) Prove that Q^1 is a scheme. (Hint: Let U_0 be the set of triples $(\mathscr{L}, x, y)$ such that x generates $\mathscr{L}$ and let U_1 be the set of triples $(\mathscr{L}, x, y)$ such that y generates $\mathscr{L}$.)

Exercise 16.5. Prove that $Q^1 \simeq \mathbf{P}^1$. (Hint: What is the intersection of U_0 and U_1? Suggestion: Use symbols $\mathbf{U}_0$ and $\mathbf{U}_1$ for the standard charts of $\mathbf{P}^1$ from Section 1.1.)

[2] This means that if $z \in \Gamma(U, \mathscr{L})$ then there is a cover of U by open subsets V such that $z\big|_V = ax\big|_V + by\big|_V$ for some $a, b \in \Gamma(V, \mathcal{O}_S)$. In other words, $\mathscr{L}$ is the smallest $\mathcal{O}_S$-submodule of itself that contains both x and y.

16.1 The tautological line bundle

If S is a scheme then a map $S \to \mathbf{P}^n$ corresponds to a linear embedding of a line bundle $L \subset \mathbf{A}^{n+1}_S$ or to a surjection $\mathscr{O}^{n+1}_S \to \mathscr{L}$ onto an invertible sheaf. In particlar, the identity map $\mathbf{P}^n \to \mathbf{P}^n$ gives

$$L \subset \mathbf{A}^{n+1}_{\mathbf{P}^n}$$
$$\mathscr{O}_{\mathbf{P}^n} \to \mathscr{L}.$$

The quotient $\mathscr{L}$ is usually denoted $\mathbf{O}_{\mathbf{P}^n}(1)$ and is called the *tautological (quotient) sheaf.* The subbundle L is called the *tautological line (sub)bundle* and is sometimes denoted $\mathscr{O}_{\mathbf{P}^n}(-1)$ by people who are sloppy about the distinction between quasicoherent sheaves and schemes of modules.

Exercise 16.6. Suppose that $f : S \to \mathbf{P}^n$ corresponds to $(\mathscr{L}, \xi_0, \ldots, \xi_n)$. Show that $f^* \mathscr{O}_{\mathbf{P}^n}(1) = \mathscr{L}$ in a canonical way.

Exercise 16.7. Show that $\mathscr{O}_{\mathbf{P}^n}(1)$ is not isomorphic to $\mathscr{O}_{\mathbf{P}^n}$. (Hint: Let A be a commutative ring, like $\mathbf{Z}[\sqrt{-5}]$, that is not a principal ideal domain and let I be a nonprincipal ideal with 2 generators. Use these to construct a map $f : \operatorname{Spec} A \to \mathbf{P}^n$ and show that $f^* \mathscr{O}_{\mathbf{P}^n}(1)$ is not isomorphic to $\mathscr{O}_{\operatorname{Spec} A}$.)

B Projective space and the Grassmannian

B.1 Projective space

For any scheme T, define $P^n(T)$ to be the set of closed embeddings of vector bundles $L \to \mathbf{A}^{n+1}_T$, where L is a line bundle over T.

Define $Q^n(T)$ to be the set of surjections of $\mathscr{O}_T$-modules $\mathscr{O}^{n+1}_T \to \mathscr{L}$ where $\mathscr{L}$ is locally free of rank 1.

Exercise B.1. Prove that P^n is isomorphic to Q^n for all n. (Hint: What you need to show here is that if $L \to \mathbf{A}^{n+1}_T$ corresponds to $\mathscr{O}^{n+1}_T \to \mathscr{L}$ then the former is a closed embedding if and only if the latter is a surjection. It's enough to prove this locally in T, so you can assume $L = \mathbf{A}^1_T$ and $\mathscr{L} = \mathscr{O}_T$.)

Do one of the following two exercises. They are two perspectives on the same thing:

Exercise B.2. Show that P^n is a scheme:

(i) Show that P^n is a Zariski sheaf.

(ii) For each $i = 0, \ldots, n$, let U_i be the subfunctor of Q^n consisting of those linear closed embeddings $f : L \to \mathbf{A}^{n+1}_T$ such that if $p_i : \mathbf{A}^{n+1}_T \to \mathbf{A}^1_T$ is the i-th projection the map $f \circ i$ is an isomorphism. Show that $U_i \simeq \mathbf{A}^n$.

(iii) Show that each U_i is representable by $\mathbf{A}^n$.

(iv) Show that the U_i cover Q^n.

Exercise B.3. Show that Q^n is a scheme:

(i) Show that Q^n is a Zariski sheaf.

(ii) For each $i = 0, \ldots, n$, let U_i be the subfunctor of Q^n consisting of those surjections $\mathcal{O}_T^{n+1} \to \mathscr{L}$ such that $\mathcal{O}_T e_i$ surjects onto $\mathscr{L}$. Show that U_i is an open subfunctor of Q^n.

(iii) Show that each U_i is representable by $\mathbf{A}^n$.

(iv) Show that the U_i cover Q^n.

Exercise B.4. Recall that we defined $\mathbf{P}^n$ previously to be $\operatorname{Proj} \mathbf{Z}[x_0, \ldots, x_n]$. Prove that $\mathbf{P}^n \simeq P^n$ or $\mathbf{P}^n \simeq Q^n$.

B.2 The Grassmannian

Fix a non-negative integer r and regard $\mathbf{A}^r$ as a vector space. That is, remember that we can use functions in $\Gamma(S, \mathcal{O}_S) = \operatorname{Hom}_{\mathbf{Sch}}(S, \mathbf{A}^1)$ to act on $\operatorname{Hom}_{\mathbf{Sch}}(S, \mathbf{A}^r)$. Define a functor

$$G : \mathbf{Sch}^\circ \to \mathbf{Sets}$$

by taking $G(S)$ to be the set of closed vector bundle subschemes $W \subset \mathbf{A}^r \times S$.

Exercise B.5. Show that G is the disjoint union of open subfunctors $\coprod_{k=0}^r G_k$ where G_k parameterizes closed vector bundle subschemes $W \subset \mathbf{A}^r \times S$ of rank k.

Correction: $\mathcal{O}_S^{\oplus k}$ was supposed to be $\mathcal{O}_S^{\oplus r}$. Thanks to John Willis.

Exercise B.6. Show that G_k is isomorphic to the functor $Q_k : \mathbf{Sch}^\circ \to \mathbf{Sets}$ where $Q_k(S)$ is the set of isomorphism classes of surjections $\mathcal{O}_S^{\oplus r} \to \mathscr{W}$, with $\mathscr{W}$ being a locally free sheaf of $\mathcal{O}_S$-modules of rank k.

Exercise B.7. Show that the functors Q_k are representable by schemes. (Hint: Use the fact that you can glue vector bundles and homomorphisms of vector bundles to prove that Q_k is a Zariski sheaf. To get an open cover observe that at each point of Q_k there is some k-element subset $I \subset \{1, \ldots, r\}$ such that $\mathcal{O}_S^{\oplus I} \to \mathscr{W}$ is surjective. Let $U_I \subset G_k$ be the subfunctor parameterizing surjections $\mathcal{O}_S^{\oplus r} \to \mathscr{W}$ such that $\mathcal{O}_S^{\oplus I} \to \mathscr{W}$ is surjective. Show that U_I is an open subfunctor of G_k and that U_I is representable by $\mathbf{A}^{k \times (r-k)}$.)

The scheme representing G_k is denoted $\mathbf{Grass}(k, r)$ and called the *Grassmannian.*

Exercise B.8. Define a functor on S-schemes parameterizing closed linear subschemes of a vector bundle V over S. Show that this is representable by an S-scheme. (Hint: After defining the functor and showing it is a sheaf, reduce to the case considered above by passing to an cover of S by open subsets U such that $V\big|_U \simeq U \times \mathbf{A}^r$.)

17 Coherent schemes

17.1 The diagonal

Exercise 17.1. The equalizer of a pair of morphisms of schemes is locally closed in the domain.

Exercise 17.2. Show that the equalizer of a pair of maps $X \rightrightarrows Y$ can be interpreted as the fiber product $X \times_{Y \times Y} \Delta Y$.

17.2 Quasicompact and quasiseparated morphisms

Reading 17.3. [Har77, Exercises 2.13, 3.2], [Vak14, §§3.6.5, 5.1, 10.1.9–12]

Recall that a scheme X is quasicompact if every open subcover of X has a finite subcover.

Definition 17.4. A morphism of schemes $f : X \to Y$ is said to be *quasicompact* if for, for any morphism of schemes $Z \to Y$ with Z quasicompact, the fiber product $Z \times_Y X$ is quasicompact.

It is said to be *quasiseparated* if for every pair of maps $g, h : Z \to X$ with Z quasicompact such that $fg = fh$, the equalizer $W \subset Z$ of g and h in Z is quasicompact.

Morphisms that are both quasicompact and quasiseparated are sometimes called *coherent*.

Exercise 17.5. Show that a morphism of schemes $f : X \to Y$ is quasiseparated if and only if the diagonal map $X \to X \times_Y X$ is quasicompact.

Exercise 17.6. Suppose that $f : X \to Y$ is a morphism of schemes such that for any quasicompact *open subset* $U \subset Y$ the preimage $f^{-1}U$ is also quasicompact. Show that f is quasicompact. (Hint: In the notation of Definition 17.4, reduce to the case where Z and Y are affine.)

17.3 Pushforward of quasicoherent sheaves

Exercise 17.7. Show that a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on $X = \operatorname{Spec} A$ is quasicoherent if and only if $\Gamma(D(f), \mathscr{F}) = \Gamma(X, \mathscr{F})_f$ for all $f \in A$.

Exercise 17.8. Show that the kernel and cokernel of a homomorphism of quasicoherent sheaves are quasicoherent sheaves.

Theorem 17.9. *Suppose that $\pi : X \to Y$ is a quasicompact and quasiseparated morphism of schemes and $\mathscr{F}$ is a quasicoherent sheaf on X. Prove that $\pi_* \mathscr{F}$ is a quasicoherent sheaf on Y.*

Exercise 17.10.
(i) Show by example that an infinite product of quasicoherent sheaves is not necessarily quasicoherent. (Hint: Use the failure of localization to commute with infinite products.)

(ii) Show by example that an infinite intersection of a quasicoherent subsheaves of a quasicoherent sheaf is not necessarily quasicoherent.

Exercise 17.11. Use the previous exercise to show that both the hypothesis of quasicompactness and quasiseparation are necessary in Theorem 17.9.

Chapter 7

Essential properties of schemes

18 Finite presentation

18.1 Filtered diagrams

Definition 18.1. A category P is said to be *filtered* if every finite diagram in P has an upper bound.

In practical terms, the definition means the following:

(i) for any pair of objects $x, y \in P$ there is an object $z \in P$ and morphisms $x \to z$ and $y \to z$;

(ii) for any pair of morphisms $x \rightrightarrows y$ in P there is a morphism $y \to z$ in P that coequalizes them.

Note that the second condition holds vacuously for a partially ordered set.[1]

18.2 Remarks on compactness

Exercise 18.2. (i) Suppose that X is a quasicompact topological space and $Y = \bigcup Y_i$ is a filtered union of open subsets. Show that any morphism $X \to Y$ factors through one of the Y_i.

(ii) Suppose that X is a quasicompact and quasiseparated topological space with a basis of quasicompact open subsets and $Y = \varinjlim Y_i$ is a filtered colimit of a diagram of topological spaces where the transition maps are open embeddings and the Y_i are all étale over Y. Show that any morphism $X \to Y$ factors through one of the Y_i.

18.3 Finite type and finite presentation

Reading 18.3. [GD67, IV.8.14]

[1] The conditions above are usually taken as the definition of a filtered diagram (and the first as the condition of a filtered partially ordered set). However, just as the first condition does not extend trivially, these two conditions do not extend trivially to higher categories. Definition 18.1 does.

Definition 18.4. A morphism of schemes $f : X \to Y$ is said, respectively, to be *locally of finite type* or *locally of finite presentation* if there is an open cover of Y by open affine subsets $V = \operatorname{Spec} A$ such that $f^{-1}V$ is covered by open affines $U = \operatorname{Spec} B$ where B is a finite type or finitely presented A-algebra. The morphism is of *finite type* if it is quasicompact and locally of finite type. It is *finitely presented* if it is locally of finite presentation and quasicompact and quasiseparated.

Exercise 18.5. Suppose that B is an A-algebra and $C = \varinjlim C_i$ is a filtered direct limit of A-algebras. Consider the map

$$\Phi : \varinjlim \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, C_i) \to \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, C).$$

(i) Prove that Φ is an injection for all $C = \varinjlim C_i$ if B is of finite type over A.

(ii) Prove that Φ is a bijection for all filtered unions $C = \bigcup C_i$ if and only if B is of finite type over A.

(iii) Prove that Φ is a bijection for all $C = \varinjlim C_i$ if and only if B is of finite presentation over A.

Should be easy

Exercise 18.6. Show that a morphism of locally noetherian schemes is of locally finite type if and only if it is of locally finite presentation.

Lemma 18.7. *Let X be an A-scheme and let $C = \varinjlim C_i$ be a colimit of A-algebras. Consider the map*

$$\Phi : \varinjlim X(C_i) \to X(C)$$

(i) If C is the filtered colimit of the C_i and X is of finite type the Φ is an injection.

(ii) If C is the filtered union of the C_i and C is an integral domain *and X is of finite type then Φ is a bijection.*

(iii) If C is the filtered colimit of the C_i and X is of finite presentation then Φ is a bijection.

Theorem 18.8. *For any scheme X, the following conditions are equivalent:*

(i) X is locally of finite presentation;

(ii) for any filtered system of commutative rings A_i with $\varinjlim A_i = A$, the map $\varinjlim X(A_i) \to X(A)$.

19 Separated and proper morphisms I

Reading 19.1. [Vak14, §§10.1, 10.3, 12.7] [Har77, §II.4]

In this section and the next we will investigate the algebro-geometric analogues of compact and Hausdorff topological spaces. Recall that a topological space X is called *Hausdorff* if, for any pair of points x and y, there are open neighborhoods $x \in U$ and $y \in V$ with $U \cap V = \varnothing$. Equivalently, $U \times V$ is an open neighborhood of (x, y) in $X \times X$ that does not meet the diagonal $X \subset X \times X$. In other words, the diagonal is closed.

Exercise 19.2. Show that the following conditions are equivalent for a topological space X:

(i) X is Hausdorff;

(ii) the diagonal $X \to X \times X$ is a closed embedding;

(iii) for any pair of maps $Z \rightrightarrows X$, their equalizer is closed in Z.

The product of two schemes does not have the product topology, so these conditions are not equivalent for schemes. We know that essentially no scheme is Hausdorff in the literal sense, but the latter two conditions still make sense. We will use these as the definition of a *separated scheme*.

A second interpretation of the Hausdorff condition is that a sequence should have at most one limit. Again, it is hard to make sense of this literally for schemes, but we can reinterpret it in a way that does make sense. Instead of sequences, we look at maps from open curves into X and stipulate that such a map can be completed in at most one way.

A first candidate for such a definition is that any map $\mathbf{A}^1_k \smallsetminus \{0\} \to X$ can be completed in at most one way to $\mathbf{A}^1_k \to X$. Indeed, if X is separated, this must be true. However, there are a few problems owing to the rigidity of algebraic geometry. There are many different kinds of open arcs, of which the above is just one. In order to get a sufficiently large list, we look at *valuation rings*.

19.1 A criterion for closed subsets

Exercise 19.3 (Repeat of Exercise 7.26)**.** Show that a closed subset of a scheme is closed under specialization but that a subset closed under specialization is not necessarily closed.

The following theorem says that, Exercise 19.3 notwithstanding, being closed under specialization *is* equivalent to being closed in most situations that arise in practice:

Theorem 19.4 ([Sta15, Tags 00HY and 01K9], [GD67, Proposition (II.7.2.1)])**.** *The image of a quasicompact morphism is closed if and only if it is stable under specialization.*

Repeat of Exercise 7.11.

Exercise 19.4.1. Let X be a scheme and Z a closed subset. Let $i : Z \to X$ be the inclusion. Define $\mathscr{A}(U)$ to be the quotient of $\mathcal{O}_X(U)$ by the relation $f \sim g$ if $f(p) = g(p)$ for all $p \in U \cap Z$. Set $\mathcal{O}_Z = i^{-1}\mathscr{A}$. Show that $(Z, \mathcal{O}_Z)$ is a reduced scheme. This is called the *reduced scheme structure* on Z.

Corollary 19.4.2. *A quasicompact morphism of schemes $f : X \to Y$ is closed if and only if specializations lift along f.*

19.2 Valuation rings

Reading 19.5. [AM69, pp. 65–67], [GD67, §II.7.1]

Definition 19.6. A *valuation ring* is an integral domain A such that for all nonzero x in the field of fractions of A, either $x \in A$ or $x^{-1} \in A$.

Exercise 19.7. (i) Show that $\mathbf{Z}_{(p)}$ is a valuation ring.

(ii) Show that $k[t]_{\mathfrak{p}}$ is a valuation ring when k is a field and $\mathfrak{p}$ is any ideal other than the zero ideal.

(iii) Show that $k[[t]]$ is a valuation ring when k is a field.

(iv) Let k be a field and let $A = \bigcup_{n\to\infty} k[[t^{1/n}]]$ be the ring of *Puiseux series*. Show that A is a valuation ring.

(v) Give an example of a local ring that is not a valuation ring.

These are a few basic facts about valuation rings. They aren't essential but they may help build intuition.

Exercise 19.8.

(i) If A is a valuation ring then the fractional ideals of A are totally ordered under inclusion.

(ii) A valuation ring is a local ring.

(iii) If A is a valuation ring then all finitely generated ideals of A are principal. (Note that this does not mean A is a principal ideal domain!)

(iv) If A is a valuation ring then the *nonzero* fractional ideals[2] of A form a group under multiplication.

(v) [AM69, Chapter 5, Exercise 30] Let K be the field of fractions of a valuation ring A. Let $v(x) = Ax$ for any $x \in K^*$. This gives a homomorphism from K^* into the group of nonzero fractional ideals of K. Show that $v(x+y) \geq \min\{v(x), v(y)\}$ where the nonzero fractional ideals are ordered by inclusion. Thus v is a *valuation*.

Theorem 19.9. *Let $x \in X(K)$ be a K-point of X and suppose that $x \rightsquigarrow y$.*[3] *Then there is a valuation ring R with field of fractions K and a map $\operatorname{Spec} R \to X$ sending the closed point of $\operatorname{Spec} R$ to y and restricting to x on the generic point.*

Thus the inclusion of a valuation ring in its field of fractions is the 'universal specialization'.

19.3 Separatedness

Definition 19.10 (Separatedness)**.** A morphism of schemes $\pi : X \to Y$ is *separated* if, for any $f, g : Z \to X$ such that $\pi f = \pi g$, the equalizer of f and g is a closed subscheme of Z.

Exercise 19.11. Let k be a field and let $X = \mathbf{A}^1_k \cup_{\mathbf{A}^1_k \smallsetminus \{0\}} \mathbf{A}^1_k$ be the affine line with its origin doubled. Show that $X \to \operatorname{Spec} \mathbf{k}$ is not separated.

Important and easy; correction: 'closed' corrected to 'separated' (thanks to Shawn)

Exercise 19.12. Show that $\pi : X \to Y$ is separated if and only if $\delta = (\mathrm{id}_X, \mathrm{id}_X) : X \to X \times_Y X$ is a closed embedding. This is how separatedness is usually defined.

Exercise 19.13. Prove that a topological space is Hausdorff if and only if its diagonal is a closed embedding.

Important and easy

Exercise 19.14. Prove that every affine scheme is separated.

Not directly related to this section but useful in the next exercise.

Exercise 19.15. A locally closed embedding of schemes with closed image is closed. (Hint: Use the fact that an embedding can be shown to be a closed embedding on an open cover of the codomain.)

Exercise 19.16. A morphism $\pi : X \to Y$ is separated if and only if it is quasiseparated and its diagonal is closed under specialization. (Hint: Make use of Exercise 19.15.)

[2] A fractional ideal is a *finitely generated* submodule of the field of fractions.

[3] Properly speaking, it is the image of $\operatorname{Spec} K$ under x that specializes to y.

This gives an intuitive picture of specialization. A specialization in $X \times_Y X$ of a point x in the diagonal yields a pair of specializations $x \rightsquigarrow x'_1$ and $x \rightsquigarrow x'_2$ in X, with both projecting to the same specialization $y \rightsquigarrow y'$ of Y. If this specialization lifts then $x'_1 = x'_2$.

Theorem 19.17 (Valuative criterion for separatedness)**.** *A quasiseparated morphism of schemes $f : X \to Y$ is separated if and only if whenever R is a valuation ring with field of fractions K, a diagram* (19.1) *admits* at most one *lift.*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{19.1}$$

19.4 Properness

Definition 19.18. A morphism of schemes $f : X \to Y$ is said to be *universally closed* if, for every Y-scheme Y' the morphism $f' : X' \to Y'$ induced by base change is closed.

Exercise 19.19. Let k be a field. Show that $\mathbf{A}^1_k \to \operatorname{Spec} k$ is closed but not universally closed.

Definition 19.20 (Properness)**.** A morphism of schemes $f : X \to Y$ is *proper* if it is separated, of finite type, and universally closed.

Exercise 19.21. Suppose that $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes. Show the following:

(i) Show that all closed embeddings are proper.

(ii) If f and g are both proper the gf is proper.

(iii) [Har77, Exercise II.4.8] If g is separated and gf is proper then f is proper.

Exercise 19.22. Show that $X \to Y$ is separated if and only if $X \to X \times_Y X$ is proper.

Theorem 19.23. *Suppose that $f : X \to Y$ is quasicompact and quasiseparated. Then f is separated and universally closed if and only if every diagram* (19.2) *admits a* unique *lift.*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{19.2}$$

Corollary 19.23.1. *Suppose that $f : X \to Y$ is of finite type and is quasiseparated. Then f is proper if and only if f satisfies the right lifting property with respect to* $\operatorname{Spec} K \subset \operatorname{Spec} R$ *for every valuation ring R with field of fractions K.*

19.5 Projective schemes

Definition 19.24. A *projective scheme* is a scheme that can be embedded inside $\mathbf{P}^N$, for some integer N, as a closed subscheme.

Theorem 19.25. *Projective schemes are proper.*

Exercise 19.26 (Nakayama's lemma)**.** Let M be a finitely generated module over a local ring A with maximal ideal $\mathfrak{p}$. Show that $\mathfrak{p}M = M$ if and only if $M = 0$.

Exercise 19.27 (Support of a finitely generated module is closed)**.** Show that the support of a finitely generated module M over an affine scheme A is closed in $\operatorname{Spec} A$. (Hint: Let I be the annihilator ideal of M and show that $M \otimes_A k(p) = 0$ if and only if $p \in D(I)$.)

Exercise 19.28. Let E be a vector bundle over a scheme Y. Let $\mathbf{P}(E) : (\mathbf{Sch}/Y)^\circ \to \mathbf{Sets}$ be functor sending an Y-scheme X to the set of closed embeddings $L \to E\big|_X$ in which L is a line bundle over X. Verify that $\mathbf{P}(E)$ is representable by a scheme. (Hint: Cover X by open sets U where $E\big|_U$ is trivial and use $\mathbf{P}^N \times U$.)

Definition 19.29 ([GD67, Proposition (II.5.5.1) and Définition (II.5.5.2)])**.** A morphism of schemes $f : X \to Y$ is said to be *projective* if there is a closed embedding $X \to \mathbf{P}(E)$ over Y, for some vector bundle E over Y.

Exercise 19.30. Prove that projective morphisms are proper.

20 Separated and proper morphisms II

Chapter 8

Étale morphisms

21 Separated and proper morphisms III

22 Étale morphisms I

Recall that a morphism of topological spaces $X \to Y$ is said to be *étale* if it is a local homeomorphism. This definition does not work well for schemes, where the Zariski topology is too coarse to detect maps that should be considered local homeomorphisms.

If you have studied differential geometry, this exercise should be essentially immediate. If you have not studied differential geometry, there is no reason to do this exercise.

Exercise 22.1. Show that a morphism of differentiable manifolds $f : X \to Y$ is a local diffeomorphism near a point x if and only if the map $df : T_x X \to T_{f(x)} Y$ is an isomorphism. (Hint: Inverse function theorem.)

Exercise 22.2. (i) Show that the map $\mathbf{C}^* \to \mathbf{C}^*$ sending z to z^n is a local homeomorphism for all nonzero $n \in \mathbf{Z}$.

(ii) Show that the map $\operatorname{Spec} \mathbf{C}[t, t^{-1}] \to \operatorname{Spec} \mathbf{C}[s, s^{-1}]$ sending s to t^n is not a local homeomorphism for any n except ± 1. (Hint: Consider the map on generic points.)

23 Étale morphisms II

Instead of a topological characterization of étale maps, we will use a geometric one. In a sense, a map of topological spaces is a local homeomorphism if its source and target are *locally indistinguishable.* Taking this as our cue, we call a map of schemes étale if its source and target are *infinitesimally indistinguishable.*

Exercise 23.1. Let $i : Z \to Z'$ be a closed embedding. Let I be the kernel of $\mathcal{O}_{Z'} \to i_* \mathcal{O}_Z$ (as a homomorphism of $\mathcal{O}_{Z'}$-modules). Show that I is a quasicoherent sheaf.

Definition 23.2. A morphism of schemes $Z \to Z'$ is said to be an *infinitesimal extension* or a *nilpotent thickening* or a *nilpotent extension* if it is a closed embedding and the sheaf of ideals $I_{Z/Z'}$ is nilpotent.

If $I_{Z/Z'}^2 = 0$ then $Z \subset Z'$ is said to be a *square-zero extension* or *square-zero thickening.*

Exercise 23.3. Show that a closed embedding $Z \to Z'$ is an infinitesimal extension if and only if there is a positive integer n and local charts $\operatorname{Spec} A \to \operatorname{Spec} A'$ for $Z \to Z'$ such that, when I is defined to be $\ker(A' \to A)$, we have $I^n = 0$.

Exercise 23.4. Show that every nilpotent thickening can be factored into a sequence of square-zero thickenings. (Hint: Take the closed subschemes defined by $I^n_{Z/Z'}$.)

Exercise 23.5. Show that if $Z \subset Z'$ is an infintesimal thickening then the inclusion of topological spaces $|Z| \subset |Z'|$ is a bijection.

Definition 23.6. A morphism of schemes $f : X \to Y$ is said to be *formally étale* if, whenever $Z \subset Z'$ is an infinitesimal thickening, any diagram of solid arrows (23.1) can be completed by a dashed arrow in a unique way.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{23.1}$$

If f is also locally of finite presentation then we say f is *étale*.

Exercise 23.7. Show that all open embeddings are étale. In a sense this shows that 'locally indistinguishable' implies 'infintiesimally indistinguishable'. (It is possible to do this directly, but you might find this exercise easier using the results from the next one.)

Correction: "open cover" in the first part changed to "basis of open subsets". Thanks to Paul.

Exercise 23.8.

(i) Show that to prove a diagram (23.1) has a unique lift, it is sufficient produce unique lifts over a basis of open subsets of Z'. (Hint: Use the fact that X and Y are Zariski sheaves.)

(ii) Show that we would have arrived at an equivalent definition of étale morphisms if we had only required liftings with respect to infinitesimal extensions of *affine schemes*.

(iii) Show that we would have arrived at an equivalent definition of étale morphisms if we had only required liftings with respect to square-zero extensions of affine schemes.

Exercise 23.9. (i) Show that the map

$$\operatorname{Spec} \mathbf{C}[t, t^{-1}] \to \operatorname{Spec} \mathbf{C}[s, s^{-1}]$$

sending s to t^n is étale for all $n \neq 0$.

(ii) Suppose k is a field of characteristic p. For which values of n is the map

$$\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[s, s^{-1}]$$

étale?

Chapter 9

Smooth morphisms

24 Étale morphisms III

24.1 The module of relative differentials

Definition 24.1. Let A be a commutative ring, let B be a commutative A-algebra, and let J be a B-module. An *A-derivation from B into J* is a function $\delta : B \to J$ such that

Der1 $\delta(A) = 0$ and

Der2 $\delta(xy) = x\delta(y) + y\delta(x)$ for all $x, y \in B$.

The set of A-derivations from B into J is denoted $\mathrm{Der}_A(B, J)$.

Exercise 24.2. Show that $\mathrm{Der}_A(B, J)$ is naturally equipped with the structure of an A-module via $(a\delta)(x) = a\delta(x)$.

Exercise 24.3. Let $B + \epsilon J$ be the commutative ring whose elements are symbols $x + \epsilon y$ with $x \in B$ and $y \in J$ with the addition rules

$$(x + \epsilon y) + (x' + \epsilon y') = (x + x') + \epsilon(y + y')$$
$$(x + \epsilon y)(x' + \epsilon y') = xx' + \epsilon(xy' + x'y).$$

(i) Show that there is a homomorphism $p : B + \epsilon J \to B$ defined by $p(x + \epsilon y) = x$.

(ii) Show that there is a homomorphism $i : B \to B + \epsilon J$ defined by $i(x) = x + \epsilon 0$.

(iii) Suppose that $f : B \to B + \epsilon J$ is an A-algebra homomorphism such that $pf = \mathrm{id}_B$. Show that $f - i$ factors through $\epsilon J \subset B + \epsilon J$ and that regarded as a map $B \to J$ it is a derivation.

(iv) Suppose that $\delta : B \to J$ is a derivation. Show that $\mathrm{id}_B + \epsilon\delta : B \to B + \epsilon J$ is a homomorphism of A-algebras.

(v) Conclude that $\mathrm{Der}_A(B, J) = \mathrm{Hom}_A^B(B, B + \epsilon J)$ (where it's your job to figure out what the notation Hom_A^B means).

Exercise 24.4. Show that there is a universal B-module $\Omega_{B/A}$ and A-derivation $d : B \to \Omega_{B/A}$. (In other words, show that the functor $J \mapsto \mathrm{Der}_A(B, J)$ is representable by a B-module $\Omega_{B/A}$.)

Definition 24.5. The universal A-derivation $B \to \Omega_{B/A}$ constructed in Exercise 24.4 is called the *module of relative differentials of B over A* or the *module of relative Kähler differentials*.

Exercise 24.6. Compute $\Omega_{B/A}$ when $B = A[x_1, \dots, x_n]$ is a polynomial ring.

Exercise 24.7 ([Har77, Proposition II.8.1], [Vak14, Theorem 21.2.9])**.** Suppose $A \to B \to C$ are homomorphisms of commutative rings.

(i) Show that for any C-module J there is a natural exact sequence of C-modules:

$$0 \to \mathrm{Der}_B(C, J) \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J)$$

Typo corrected in the exact sequence. Tensor product is over B not over A. Thanks Ryan.

(ii) Deduce an exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

(iii) Find an example to show that the sequence can't be completed with a $0 \to C \otimes_A \Omega_{B/A}$ on the left. (Hint: Consider $A = k$ a field, $B = k[x]/(x^2)$, and $C = B/xB \simeq k$.)

Exercise 24.8. Suppose that $B \to C$ is an *epimorphism* of A-algebras.[1]

(i) Show that $\Omega_{C/B} = 0$. (This isn't used in the rest of the exercise.)

(ii) Let I be the kernel of $B \to C$. For any C-module J, construct an exact sequence:

$$0 \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J) \to \mathrm{Hom}_{B\text{-}\mathbf{Mod}}(I, J)$$

(iii) Conclude that there is an exact sequence of C-modules:

$$I/I^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$$

(Hint: $I/I^2 \simeq C \otimes_B I$. Why?)

(iv) Show by example that the sequence can't be completed by $0 \to I/I^2$ on the left and remain exact. (Hint: Consider $A = k$ a field, $B = k[x]$, and $C = B/x^2B = k[x]/(x^2)$.)

25 Étale morphisms IV

25.1 Extensions of algebras

Extensions of algebras mean square-zero extensions. Thanks Ryan.

Definition 25.1. Let A be a commutative ring, B a A-algebra, and J a B-module. An A-algebra extension of B by J is a surjective homomorphism *with square-zero kernel* of A-algebras $B' \to B$ and an identification of the kernel of this surjection with J. A morphism from an extension B' to an extension B'' is a homomorphism of A-algebras that induces

[1] This means that $\mathrm{Hom}_{A\text{-}\mathbf{Alg}}(C, D) \to \mathrm{Hom}_{A\text{-}\mathbf{Alg}}(B, D)$ is injective for any A-algebra D. This includes surjections and localizations.

the identity on J and induces the identity modulo J. In other words, it is a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J \arrow[r] & B'' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

The isomorphism classes of A-algebra extensions of B by J are denoted $\mathrm{Exal}_A(B, J)$.

Exercise 25.2. (i) Show that the automorphism group of $B + \epsilon J$ as an A-algebra extension of B is $\mathrm{Der}_A(B, J)$. Conclude that A-algebra extensions can have nonzero automorphisms.

(ii) Show that every morphism of A-algebra extensions is an isomorphism.

(iii) Construct a bijection between the isomorphisms $B' \simeq B + \epsilon J$ and the A-algebra sections of $B' \to B$.

Exercise 25.3. Let $q : \widetilde{A} \to B$ be a surjection.

(i) Find an identification between $\mathrm{Exal}_{\widetilde{A}}(B, J)$ and

$$\mathrm{Hom}_{\widetilde{A}\text{-}\mathbf{Alg}}(I_{B/\widetilde{A}}, J) = \mathrm{Hom}_{B\text{-}\mathbf{Mod}}(B \otimes_{\widetilde{A}} I_{B/\widetilde{A}}, J).$$

(ii) Show that under this identification, the zero element corresponds to $B' = B + \epsilon J$ with the $\widetilde{A}$-algebra structure coming from the homomorphism $q + 0\epsilon$. Show that, up to isomorphism, this is the only $\widetilde{A}$-algebra extension $B' \to B$ that has a section by a $\widetilde{A}$-algebra homomorphism.

25.2 An algebraic characterization of étale morphisms

Definition 25.4. Suppose that B is an A-algebra. Let $\widetilde{A} \to B$ be a surjection of A-algebras, where $\widetilde{A}$ is a polynomial ring over A. The *(truncated) cotangent complex* of B over A is the 2-term complex (with respect to this presentation) is the complex

$$B \otimes_{\widetilde{A}} I_{B/\widetilde{A}} \xrightarrow{d} B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}.$$

The map sends $b \otimes f$ to $b \otimes df$. The truncated cotangent complex is denoted $\tau_{\geq -1}\mathbf{L}_{B/A}$.

Exercise 25.5. Show that, up to quasi-isomorphism, $\tau_{\geq -1}\mathbf{L}_{B/A}$ is independent of $\widetilde{A}$.

Theorem 25.6. *A map of affine schemes* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *is étale if and only if*

$$d : B \otimes_{\widetilde{A}} I_{B/\widetilde{A}} \to B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}$$

is an isomorphism.

Consider an extension problem in which C' is a square-zero extension of C by the ideal J:

$$\begin{tikzcd} C & B \arrow[l] \arrow[dl, dashed] \\ C' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{25.1}$$

Exercise 25.7. Show that solving the lifting problem (25.1) is equivalent to solving the lifting problem below, in which $B' = C' \times_C B$:

$$\begin{tikzcd} B \arrow[r, equal] & B \arrow[dl, dashed] \\ B' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{25.2}$$

Exercise 25.8. Show that B is formally étale over A if and only if $\mathrm{Der}_A(B, J) = \mathrm{Exal}_A(B, J) = 0$.

Two typos corrected here. The target of the map is $\mathrm{Exal}_{\widetilde{A}}(B, J)$ and the ideal is $I_{B/\widetilde{A}}$. Thanks to Ryan for catching them.

Exercise 25.9. Let $\widetilde{A} \to B$ be any surjection. Construct a map

$$\mathrm{Der}_A(\widetilde{A}, J) \to \mathrm{Exal}_{\widetilde{A}}(B, J)$$

and identify it with the map

$$\mathrm{Hom}_{B\text{-}\mathbf{Mod}}(B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}, J) \to \mathrm{Hom}_{B\text{-}\mathbf{Mod}}(B \otimes_{\widetilde{A}} I_{B/\widetilde{A}}, J). \tag{25.3}$$

Exercise 25.10. (i) Suppose that $\widetilde{A} \to B$ is a surjection of A-algebras. Construct a commutative diagram in which the long row is exact and the morphism in the second row is induced by $d : B \otimes_{\widetilde{A}} I_{B/\widetilde{A}} \to B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}_A(B, J) \arrow[r] & \mathrm{Der}_A(\widetilde{A}, J) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Exal}_{\widetilde{A}}(B, J) \arrow[r] \arrow[d, "\wr"] & \mathrm{Exal}_A(B, J) \\ & & \mathrm{Hom}(B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}, J) \arrow[r] & \mathrm{Hom}(B \otimes_{\widetilde{A}} I_{B/\widetilde{A}}, J) & \end{tikzcd}$$

(ii) Show that if $\widetilde{A}$ is a *free* A-algebra then the map

$$\mathrm{Exal}_{\widetilde{A}}(B, J) \to \mathrm{Exal}_A(B, J)$$

is surjective.

(iii) Prove that

$$d : B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A} \to B \otimes_{\widetilde{A}} I_{\widetilde{A}/A}$$

is an isomorphism if and only if B is formally étale over A.

25.3 A differential characterization of étale morphisms

Exercise 25.11. Let $\widetilde{A} = A[x_1, \ldots, x_n]$ and let $I = (f_1, \ldots, f_m)$.

(i) Show that $B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A} = \sum B dx_i$.

(ii) Show that $B \otimes_{\widetilde{A}} I$ is generated by $f_1, \ldots, f_m$.

(iii) Show that the map

$$\sum Bf_i \to B \otimes_{\widetilde{A}} I \to B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A} = \sum B dx_i$$

is given by the following $n \times m$ matrix:

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

(iv) Under the assumption $m = n$, conclude that $d : B \otimes_{\widetilde{A}} I \to B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}$ is an isomorphism if and only if $\det \mathcal{J} \in B^*$.

Exercise 25.12. Prove that $\operatorname{Spec} k[t, t^{-1}] \to k[s, s^{-1}]$, given by $s \mapsto t^n$, is étale if and only if the characteristic of k does not divide n. (Hint: Identify $k[t, t^{-1}] = k[s, s^{-1}, t]/(t^n - s)$ and use the differential criterion.)

C Bézout's theorem

Theorem C.1. *If C and D are algebraic curves in $\mathbf{A}^2_k$ that meet transversally and do not meet at infinity then $|(C \cap D)(k)| = \deg(C)\deg(D)$ for any algebraically closed field k.*

Consider the moduli space of all such polynomials, $\mathbf{A}^N = \operatorname{Spec} A$ where $N = \binom{d+2}{d} + \binom{e+2}{e}$. Let $X \subset \mathbf{A}^N \times \mathbf{A}^2$ be the locus of (f, g, p) such that $f(p) = g(p) = 0$. Let $\pi : X \to \mathbf{A}^N$ be the projection.

Note that $C = V(f)$ and $D = V(g)$ *meet transversally* if and only if the fiber of X over the map $(f, g) : \operatorname{Spec} k \to \mathbf{A}^N$ is étale over $\operatorname{Spec} k$.

Exercise C.2. If

$$\begin{aligned} f &= (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_d) \\ g &= (y - \beta_1)(y - \beta_2) \cdots (y - \beta_e) \end{aligned}$$

then $V(f, g)$ consists of de reduced points.

Exercise C.3. There is a non-empty open subset U of $\mathbf{A}^N$ such that $\pi^{-1}U$ is étale over U.

Exercise C.4. Show that there is a non-empty open subset of $\mathbf{A}^N$ over which X contains no points at infinity. Show that X is proper over this open subset.

Exercise C.5. Conclude that there is an open subset $U \subset \mathbf{A}^N$ containing the example from Exercise C.2 such that $p^{-1}U$ is proper and étale over U.

Exercise C.6. Show that all geometric fibers of X over U have the same number of points. (Hint: Let k be an algebraically closed field and consider a map $h : \operatorname{Spec} k[[t]] \to U$. Construct a bijection between the closed fiber of $h^{-1}X$ and the set of points of the general fiber with residue field $k((t))$ using the valuative criterion for properness and the formal criterion for étale morphisms.)

26 Smooth and unramified morphisms

Definition 26.1. A morphism of schemes $f : X \to Y$ is said to be *formally unramified* if any infinitesimal lifting problem

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has *at most one* solution. A morphsim that is formally unramified and locally of finite type is said to be *unramified*.

This definition has been changed! The infinitesimal extension is now required to be affine. This definition is equivalent to the one given earlier, but to prove the equivalence requries cohomology.

Definition 26.2. A morphism of schemes $f : X \to Y$ is said to be *formally smooth* if any infinitesimal lifting problem

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r] & Y \end{tikzcd}$$

has at least one solution *when S' is affine*. A morphism that is both formally smooth and locally of finite presentation is said to be *smooth*.

Exercise 26.3. Show that formally étale is the conjunction of formally smooth and formally unramified. (Note: This is not completely trivial! You will have to glue some morphisms.)

Exercise 26.4. (i) Suppose that $f : X \to Y$ induces an injection between functors of points. Show that f is unramified.

(ii) Conclude that locally closed embeddings are unramified.

(iii) Give an example of an unramified morphism that is not an injection on functors of points. (Hint: Consider the map $f : \mathbf{A}^1 \to \mathbf{A}^2$ given by $f(x) = (t^2 - 1, (t^2 - 1)t)$. Show that this is a closed embedding away from either of the points $t = \pm 1$.)

Exercise 26.5. (i) Show that $\mathbf{A}^n$ is smooth for all $n \geq 0$.

(ii) Show that the base change of a smooth morphism is smooth.

Deformation theory

Suppose we have a sequence of homomorphisms of commutative rings $A \xrightarrow{f} B \xrightarrow{g} C$. We saw earlier that there is an exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

One might be tempted to ask how this sequence can be extended on the left. It turns out that it is easier to consider all C-modules J and the dual sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Omega_{C/B}, J) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\Omega_{C/A}, J) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(C \otimes_B \Omega_{B/A}, J) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Der}_B(C, J) \arrow[r] & \mathrm{Der}_A(C, J) \arrow[r] & \mathrm{Der}_A(B, J)
\end{tikzcd}$$

Exercise 26.6. Show that this sequence can be continued to a 6-term sequence:

$$0 \to \mathrm{Der}_B(C, J) \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J) \to \mathrm{Exal}_A(C, J) \to \mathrm{Exal}_A(B, J)$$

Exercise 26.7. (i) Show that f is formally smooth if and only if $\mathrm{Exal}_A(B, J) = 0$ for all J.

(ii) Show that g is formally unramified if and only if $\mathrm{Der}_B(C, J) = 0$ for all J.

(iii) Assume f is formally smooth and g is formally unramified. Show that gf is formally étale if and only if

$$\mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J)$$

is an isomorphism.

Part II

General properties of schemes

Chapter 10

Dimension

27 Dimension of smooth schemes

27.1 The tangent bundle

Exercise 27.1. Suppose that A is a commutative ring and B is an A-algebra. Show that the natural map

$$B[f^{-1}] \otimes_B \Omega_{B/A} \to \Omega_{B[f^{-1}]/A}$$

is an isomorphism. (Hint: Consider the functors they represent.)

Exercise 27.2. Suppose $A \to B$ is a homomorphism of commutative rings and let $X \to Y$ be the associated morphism of affine schemes. For each principal open affine $D(f) \subset \operatorname{Spec} B$, define $\Omega_{X/Y}(D(f)) = \Omega_{B[f^{-1}]/A}$. Show that $\Omega_{X/Y}$ is a quasicoherent sheaf on X.

Exercise 27.3. Let $f : X \to Y$ be a morphism of schemes. Construct a quasicoherent sheaf $\Omega_{X/Y}$ on X such that if $U \subset X$ and $V \subset Y$ are open affines with $U \subset f^{-1}V$ we have $\Omega_{X/Y}\big|_U = \Omega_{U/V}$. (Hint: One strategy here is to glue together the constructions from the previous exercise. Another is to construct $d : \mathscr{O}_X \to \Omega_{X/Y}$ as the universal $f^{-1}\mathscr{O}_Y$-derivation. Still another is to take $\Omega_{X/Y} = \Delta^{-1}(\mathscr{I}/\mathscr{I}^2)$ where $\Delta : X \to X \times_Y X$ is the inclusion of the diagonal.)

Exercise 27.4. Let S be a scheme and let $\mathscr{J}$ be a quasicoherent sheaf on S. Define $\mathscr{O}_{S[\mathscr{J}]}(U) = \mathscr{O}_S(U) + \epsilon\mathscr{J}(U)$ for all open $U \subset S$.

(i) Show that $\mathscr{O}_{S[\mathscr{J}]}$ is the structure sheaf of a scheme $S[\mathscr{J}]$ whose underlying topological space is the same as that of S.

(ii) Construct a closed embedding $S \to S[\mathscr{J}]$ and a canonical retraction $S[\mathscr{J}] \to S$.

When $\mathscr{J} = \mathscr{O}_S$ we also write $S[\epsilon]$.

Exercise 27.5. Let $f : X \to Y$ be a morphism of schemes. Define $T_{X/Y}(S)$ to be the set of commutative diagrams

$$\begin{array}{ccc} S[\epsilon] & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ S & \longrightarrow & Y \end{array}$$

where $S[\epsilon] \to S$ is the retraction constructed in the last exercise. Show that $T_{X/Y}$ is representable by $\mathbf{V}(\Omega_{X/Y})$.

The scheme $T_{X/Y}$ constructed in the last exercise is known as the relative tangent bundle of X over Y.

27.2 Relative dimension

Theorem 27.6. *Suppose that $f : X \to Y$ is smooth. Then $T_{X/Y}$ is a vector bundle.*

Exercise 27.7. Suppose that B is a formally smooth A-algebra. Show that $\Omega_{B/A}$ is projective as a B-module.

Definition 27.8. A sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on a scheme X is said to be *locally of finite presentation* if there is a cover of X by open subschemes U such that there is a presentation

$$\mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus m} \to \mathscr{F}\big|_U \to 0$$

with both m and n finite.

Exercise 27.9. Suppose that $f : X \to Y$ is locally of finite presentation. Show that $\Omega_{X/Y}$ is locally of finite presentation.

The following exercises will now complete the proof of Theorem 27.6.

Imperative if you haven't done it before. Skip it if you have.

Exercise 27.10 (Nakayama's Lemma)**.** Suppose A is a local ring with residue field k and maximal ideal $\mathfrak{m}$ and M is a finitely generated A-module. Prove that the following conditions are equivalent:

(i) $M = 0$

(ii) $M = \mathfrak{m}M$

(iii) $M/\mathfrak{m}M = 0$

(iv) $M \otimes_A k = 0$

Exercise 27.11. (i) Prove that a finitely presented A-module M is locally free if and only if $M_\mathfrak{p}$ is free as an $A_\mathfrak{p}$-module for every prime $\mathfrak{p}$ of A.

(ii) Prove that a finitely presented A-module M is projective if and only if $M_\mathfrak{p}$ is projective for every prime $\mathfrak{p}$ of A. (Hint: Show that $\mathrm{Hom}_{A_\mathfrak{p}\text{-}\mathbf{Mod}}(M_\mathfrak{p}, N_\mathfrak{p}) = \mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M, N)_\mathfrak{p}$. You will need the finite presentation for this.)

(iii) Prove that a finitely presented projective module over a local ring is free. (Hint: Choose generators of $M \otimes_A k$ where k is the residue field. Lift these to M and use Nakayama's lemma to conclude that these generate M. Obtain a surjection $A^n \to M$ that induces an isomorphism upon passage to the residue field. Let $N \subset A^n$ be the kernel. Use the fact that M is projective to get an isomorphism $A^n \simeq N \times M$. Conclude that $N \otimes_A k = 0$ and apply Nakayama's lemma again.)

(iv) Prove that an A-module M of finite presentation is locally free if and only if it is projective.

Definition 27.12. Suppose that $f : X \to Y$ is a smooth morphism of schemes. If $T_{X/Y}$ has rank n then we say f is *smooth of relative dimension n*.

27.3 The structure of smooth morphisms

Theorem 27.13. *Suppose that $\pi : X \to Y$ is smooth of relative dimension n. Then there is an cover of X by open subsets U such that $U \to Y$ factors as an étale map $U \to \mathbf{A}^n_Y$.*

28 Dimension I

Reading 28.1. [AM69, Chapter 11], [Vak14, Chapter 11], [GD67, IV.0.16]

We introduce several approaches to the dimension of a commutative ring. The theory works best in the case of a noetherian local ring, and we eventually define the dimension of a non-local noetherian ring to be the maximum of the dimensions of its local rings.

28.1 Chevalley dimension

Definition 28.2. If A is a noetherian local ring, an *ideal of definition* of A is an ideal $\mathfrak{q}$ whose radical is the maximal ideal of A.

The *Chevalley dimension* of A is the minimal number of generators of an ideal of definition of A.

Should be immediate.

Exercise 28.3. Show that the Chevalley dimension of a noetherian local ring A with maximal ideal $\mathfrak{m}$ is the minimal number of elements $f_1, \ldots, f_n$ of A such that $V(f_1, \ldots, f_n) = \{\mathfrak{m}\}$.

28.2 Artin–Rees lemma

This section follows [AM69, Chapter 11].

Definition 28.4. Let A be a commutative ring, $I \subset A$ an ideal, and M an A-module. A decreasing filtration of

$$M = F^0M \supset F^1M \supset F^2M \supset \cdots$$

is called an *I-filtration* if $IF^nM \subset F^{n+1}M$ for all n. It is called a *stable I-filtration* if $IF^nM = F^{n+1}M$ for all $n \gg 0$.

We are really only interested in the filtration $F^nM = I^nM$, but we run into an unfortunate difficulty. If $M' \subset M$ then $I^nM' \neq M' \cap I^nM$. That is, we get a second filtration on M' by setting $F^nM' = M' \cap I^nM$. The Artin–Rees lemma says that when A is noetherian and M is finitely generated, these two filtrations aren't that different.

Theorem 28.5 (Artin–Rees lemma)**.** *If A is noetherian and M is finitely generated, every I-filtration F of M is stable.*

Exercise 28.6. We will prove the Artin–Rees lemma using the *Rees algebra* $B = A[tI] = \sum_{n=0}^{\infty} t^n I^n$ and the modules $N = \sum_{n=0}^{\infty} t^n I^n M$ and $N' = \sum_{n=0}^{\infty} t^n F^n M$.

(i) Prove that the Rees algebra is noetherian if A is noetherian.

(ii) Prove that N' is a B-submodule of N and that N is a finitely generated B-module. Conclude that N' is finitely generated.

(iii) Prove the Artin–Rees lemma. (Hint: Choose n such that all generators of N' have degrees $\leq n$.)

28.3 Hilbert–Samuel dimension

Definition 28.7. Let A be a noetherian local ring and let M be an A-module. If

$$M = F^0M \supsetneq F^1M \supsetneq \cdots \supsetneq F^nM = 0.$$

is a maximal filtration of M, the number n is called the *length* of M and is denoted $\operatorname{length}(M)$.

Exercise 28.8. Show that the length of M is the dimension (over the residue field) of the graded module

$$\operatorname{gr}(M) = \sum_{k=0}^{\infty} \mathfrak{m}^k M/\mathfrak{m}^{k+1}M$$

where $\mathfrak{m}$ is the maximal ideal of A. Conclude that the definition of the length does not depend on the choice of filtration F.

Exercise 28.9. Show that the length is additive in short exact sequences: if

$$0 \to M' \to M \to M'' \to 0$$

is exact then $\operatorname{length}(M) = \operatorname{length}(M') + \operatorname{length}(M'')$.

Definition 28.10 (Hilbert–Samuel function)**.** Let A be a noetherian local ring and M an A-module with a descending filtration F. The *Hilbert–Samuel function* associated to F is $h(M, F, n) = \operatorname{length}(M/F^nM)$. When F is the filtration associated to an ideal I, we write $h(M, I, n)$.

The Hilbert function turns out to be a polynomial:

Exercise 28.11. Let A be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, M a finitely generated A-module, and F a descending $\mathfrak{m}$-filtration on M.

(i) Show that $h(M, F, n) = h(\operatorname{gr} M, n)$ where $\operatorname{gr} M = \sum_{n \geq 0} F^nM/F^{n+1}M$ is the associated graded ring of M, filtered by degree.

(ii) Show that $h(M, F, n)$ agrees with a polynomial for $n \gg 0$.

Definition 28.12. In view of Exercise 28.11, the Hilbert–Samuel function $h(M, F, n)$ agrees with a polynomial for large n. We call this polynomial the *Hilbert–Samuel polynomial* and notate it $P(M, F, n)$.

Exercise 28.13. Show that if $\mathfrak{q}$ and $\mathfrak{p}$ are ideals of definition of A then $P(M, \mathfrak{p})$ and $P(M, \mathfrak{q})$ have the same degree.

Exercise 28.14. Let A be a noetherian local ring, $\mathfrak{q}$ an ideal of definition, F a descending $\mathfrak{q}$-stable filtration on a finitely generated A-module M. Show that $P(M, F)$ and $P(M, \mathfrak{q})$ have the same degree and leading coefficient.

Definition 28.15. The *Hilbert–Samuel dimension* of A is the degree of $P(A, \mathfrak{m})$.

28.4 Krull dimension

Definition 28.16. The *Krull dimension* of A is the length of the longest chain of nontrivial specializations in $\operatorname{Spec} A$. Equivalently, it is the length of a maximal chain of prime ideals in A.

29 Dimension II

29.1 Equivalence

Theorem 29.1 (Krull–Chevalley–Samuel [GD67, Théorème (0.16.2.3)], [AM69, Theorem 11.4])**.** *If A is a noetherian local ring, the Krull dimension, the Hilbert–Samuel dimension, and the Chevalley dimension are all the same.*

Let $\delta(A)$ denote the Chevalley dimension, $d(A)$ the degree of the Hilbert–Samuel polynomial, and $\dim(A)$ the Krull dimension.

Not essential, as this follows from the theorem and is not needed to prove it. But it's good practice.

Exercise 29.2. Show that the following statements are all equivalent (without using Theorem 29.1):

(i) $\delta(A) = 0$;

(ii) $d(A) = 0$;

(iii) $\dim(A) = 0$;

(iv) $\mathfrak{m}$ is nilpotent.

Exercise 29.3. Show that $\dim(A) \leq d(A)$.

Exercise 29.4. Show that $d(A) \leq \delta(A)$.

Exercise 29.5 ([Vak14, Proposition 11.2.13], [Eis91, Lemma I.3.3])**.** Let $X = \operatorname{Spec} A$ be an affine scheme, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ points of X, and let I be an ideal of X with $Z = V(I)$. Assume that Z does not contain any of the $\mathfrak{p}_i$. Then there is some $f \in I$ such that $f(\mathfrak{p}_i) \neq 0$ for all i.

Exercise 29.6. Prove $\delta(A) \leq \dim(A)$.

29.2 Codimension

Definition 29.7 ([Vak14, §11.1.4])**.** The *codimension* of an *irreducible* closed subset Z of a notherian scheme X is the dimension of the local ring at the generic point of Z.

Exercise 29.8 ([Vak14, Theorems 11.3.3, 11.3.7, §11.5])**.** Prove Krull's Hauptidealsatz: Let A be a noetherian local ring and $f_1, \dots, f_n \in A$. Show that $\operatorname{codim}_X V(f_1, \dots, f_n) \leq n$. (Hint: Use Chevalley dimension.)

Exercise 29.9. (i) Prove that for any noetherian local ring A and any prime ideal $\mathfrak{p} \subset A$ we have

$$\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} \leq \dim A$$

(ii) Give an example of a noetherian local ring A and a prime ideal $\mathfrak{p} \subset A$ such that

$$\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} \neq \dim A$$

29.3 Examples

Exercise 29.10. Compute $\dim \operatorname{Spec} \mathbf{Z}$.

Exercise 29.11. Let k be a field, let $A = k[x_1, \ldots, x_n]$, and let $\mathfrak{p} \subset A$ be the ideal $(x_1, \ldots, x_n)A$. Compute $\dim A_{\mathfrak{p}}$. (Once we have proved the Nullstellensatz, this will be a calculation of the dimension of $\mathbf{A}^n_k$.)

Exercise 29.12. Compute $\dim \mathbf{A}^n$ at a closed point.

Exercise 29.13. Suppose that X is a smooth scheme over a field k. Prove that $\dim X$ (at any closed point) coincides with the rank of the tangent bundle $T_{X/\operatorname{Spec} k}$.

29.4 Regularity

Definition 29.14. A noetherian local ring A with maximal ideal $\mathfrak{m}$ is said to be *regular* if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Exercise 29.15. Suppose that X is a smooth scheme over a field k. Show that the local ring of X at any point is regular.

Exercise 29.16. Give an example of a regular scheme that is not smooth. (Hint: An inseparable field extension.)

Exercise 29.17. Show that the rank of the tangent bundle of a smooth scheme over a field coincides with the dimension of the scheme.

Chapter 11

Algebraic properties of schemes

30 Finite, quasi-finite, and integral morphisms

Definition 30.1. A morphism of schemes $f : X \to Y$ is said to be *finite* if there is a cover of Y by open affine subschemes $V = \operatorname{Spec} A$ such that $f^{-1}V = \operatorname{Spec} B$ with B finite as an A-module.

Reorganization of definition. Should be easy.

Exercise 30.2. Show that $f : X \to Y$ is finite if and only if it is affine and $f_*\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-modules of finite type.

Trivial.

Exercise 30.3. Show that closed embeddings are finite morphisms.

Definition 30.4. A morphism of commutative rings $A \to B$ is said to be a *integral* if every element of B satisfies a monic polynomial with coefficients in A.[1] A morphism of schemes $f : X \to Y$ is said to be *integral* if there is a cover of Y by open subschemes $V = \operatorname{Spec} A$ such that $f^{-1}V = \operatorname{Spec} B$ where B is an integral extension of A.

Definition 30.5. A morphism of schemes is *quasifinite* if it is of finite type and has finite fibers.

Exercise 30.6. (i) Show that finite morphisms are quasifinite.

(ii) Give an example of a quasifinite morphism that is not finite. (Hint: open embedding.)

Exercise 30.7 (Cayley–Hamilton theorem [Sta15, Tag 00DX])**.** Suppose A is a commutative ring, M is a finitely generated A-module, and f is an endomorphism of M. Then f satisfies an integral polynomial with coefficients in A. If M is free, this polynomial can be taken to be the characteristic polynomial.

(i) Reduce to the case where M is free.

(ii) Reduce to the case where A is an integral domain.

(iii) Reduce to the case where A is a field and the characteristic polynomial splits into linear factors.

[1] Note that integral morphisms of commutative rings are not necessarily injective. This confused me for a long time.

(iv) Show that M is a finite direct sum of generalized eigenspaces.[2]

(v) Show the theorem is true when f acts nilpotently on a vector space.

(vi) Conclude that the theorem is true for all the generalized eigenspaces of M and therefore for M itself.

Exercise 30.8 ([Sta15, Tag 02JJ])**.** Show that a morphism is finite if and only if it is integral and of finite type.

Exercise 30.9. Give an example of an integral extension that is not free. (Hint: Normalize a nodal or cuspidal plane curve.)

31 Integral morphisms and dimension

31.1 Lifting inclusions of primes

Exercise 31.1. Suppose $A \subset B$ is an integral *extension*. Show that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

This exercise can be used in Exercise 31.3, but so can Exercise **??**, which might be easier.

Exercise 31.2 ([AM69, Proposition 5.7])**.** Let $A \subset B$ be an integral extension of commutative rings. Then A is a field if and only if B is a field.

Exercise 31.3 ([Vak14, Theorem 7.2.5])**.** Prove that specializations lift along integral morphisms.

Exercise 31.4. Prove that integral morphisms are universally closed.

Exercise 31.5. Show that finite morphisms are proper.

Exercise 31.6 ([Vak14, Exercise 11.1.E])**.** Suppose $f : X \to Y$ is an integral *extension*. Prove that $\dim X = \dim Y$.

Exercise 31.7. Let k be a field. Prove that for any maximal ideal $\mathfrak{p}$ of $A = k[x_1, \ldots, x_n]$, we have $\dim A_{\mathfrak{p}} = n$. (Hint: Reduce to the case of an algebraically closed field k and use the Nullstellenstaz.)

Theorem 31.8. *(i) A morphism of schemes is integral if and only if it is both affine and universally closed.*

(ii) A morphism of schemes is finite if and only if it is both affine and proper.

Exercise 31.9 ([ano], [Sta15, Tag 01WM])**.** Prove the theorem using the following steps:

(i) Suppose $\varphi : A \to B$ is injective and the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is closed and $f \in A$ is an element such that $\varphi(f) \in B^*$. Show that $f \in A^*$.

(ii) Suppose $\varphi : A \to B$ is an injection such that the induced map $\operatorname{Spec} B[t] \to \operatorname{Spec} A[t]$ is closed. Then φ is integral.

(iii) Complete the proof of the theorem.

[2] The generalized ξ-eigenspace of M is the submodule $N \subset M$ containing all $x \in M$ annihilated by some power of $(f - \xi \mathrm{id})$.

31.2 Noether normalization

Theorem 31.10 (Noether normalization [Mum99, §I.1], [Vak14, 11.2.4])**.** *Suppose k is a field and B is an integral domain of finite type over k. Then there is a polynomial subring $A \subset B$ such that B is a finite extension of A (as a module).*

Corollary 31.10.1. *Suppose that B is an integral domain of finite type over a field k. Let K be the field of fractions of B. For any maximal ideal $\mathfrak{p} \subset B$, the dimension* $\dim B_{\mathfrak{p}}$ *coincides with* $\operatorname{tr.deg}_k K$.

32 Chevalley's theorem

Reading 32.1. [Vak14, §7.4],

Theorem 32.2. *Let A be a noetherian integral domain, B an A-algebra of finite type, and M is a B-module of finite type. There is a non-zero $f \in A$ such that $A[f^{-1}] \otimes_A M$ is free as a $A[f^{-1}]$-module.*

Definition 32.3 ([GD71, Définition 0.2.3.1, 0.2.3.2, 0.2.3.10])**.** An open subset U of a scheme X is said to be *retrocompact* if the inclusion $U \subset X$ is quasicompact.

Let X be an affine scheme. A subset of X is called *constructible* if it can be constructed using only the retrocompact open subsets of X and a finite process of intersections and passages to complementary subsets.

A subset Z of a scheme X is said to be *locally constructible* if there is an cover of X by affine open subschemes U such that the intersection $Z \cap U$ is a constructible subset of U.

Exercise 32.4. Show that an open subset of an affine scheme is retrocompact if and only if it is the complement of a closed subscheme of finite presentation.

Exercise 32.5. Give an example of an open subset of an affine scheme that is not a retrocompact open subset.

Exercise 32.6. Let $f : X \to Y$ be a morphism of schemes. Show that the pullback of a constructible subset of Y to X is constructible.

Exercise 32.7. Show that a locally constructible subset of an affine scheme is constructible.

Exercise 32.8. Show that a subset of a *noetherian* scheme is constructible if and only if it is a finite union of underlying subsets of locally closed subschemes.

Exercise 32.9. Let X be a scheme. For each open $U \subset X$, let $\mathscr{S}(U)$ be the collection of all subsets of U and let $\mathscr{C}(U)$ be the set of all locally constructible subsets.

(i) Show that $\mathscr{S}$ is a sheaf and $\mathscr{C}$ is a subsheaf.

(ii) Show that $\mathscr{C}$ is the smallest subsheaf of $\mathscr{S}$ such that $\mathscr{S}(U)$ includes $D(f)$ and $V(f)$ when $U = \operatorname{Spec} A$ and $f \in A$, and is stable under finite union and finite intersection.

Exercise 32.10. Show that a subset of an affine scheme X is constructible if and only if it is a finite union of sets of the form $U \cap V$ where U is a retrocompact open and V is a closed subset of finite presentation.

Theorem 32.11. *Let $f : X \to Y$ be a quasicompact morphism that is locally of finite presentation and $Z \subset X$ a constructible subset. Then $f(Z)$ is a constructible subset of Y.*

32.1 A criterion for openness

Exercise 32.12. Show that a subset of an affine scheme X is constructible if and only if it is the image of a morphism of finite presentation.

Exercise 32.13. Show that a locally constructible subset of a scheme X is open if and only if it is stable under generization.

32.2 Nullstellensatz

Exercise 32.14 ([Vak14, 7.4.3])**.** Suppose that K is a field extension of k that is finitely generated as a k-algebra. Show that K is finitely generated as a k-module.

Chapter 12

Flatness

33 Flatness I

Reading 33.1. [Vak14, Chapter 24], [Har77, §III.9]

Definition 33.2. Let A be a commutative ring. An A-module M is said to be *flat* if $N \otimes_A M$ is an exact functor of N. An A-algebra B is said to be *flat* if it is flat as an A-module.

Definition 33.3. A morphism of schemes $f : X \to Y$ is said to be *flat* if $f^* : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$ is an exact functor. More generally, a quasicoherent sheaf $\mathscr{F}$ on X is said to be *Y-flat* if $\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{G}$ is an exact functor of $\mathscr{G} \in \mathbf{QCoh}(Y)$.[1]

Exercise 33.4. Show that $f : X \to Y$ is flat if and only if there are open charts by maps $\operatorname{Spec} B \to \operatorname{Spec} A$ where B is a flat A-algebra.

Exercise 33.5 ([Har77, Proposition 9.2], [Vak14, Exercises 24.2.A, 24.2.C, 24.2.D, 24.2.E])**.**

(i) Show that open embeddings are flat.

(ii) Let k be a field. Show that all maps $X \to \operatorname{Spec} k$ are flat.

(iii) Show that the base change of a flat map is flat.

(iv) Show that $\mathbf{A}^n_Y \to Y$ is flat.

(v) Show that a composition of flat maps is flat.

33.1 Openness

Exercise 33.6. (i) Let $f : X \to Y$ be a flat morphism. Show that the image of f is stable under generization.

(ii) Flat morphisms of finite presentation are open.

[1] These definitions can be made more generally for sheaves of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules. The result is equivalent for schemes.

33.2 Generic flatness

Reading 33.7. [Vak14, §§24.5.8–24.5.13]

Theorem 33.8. *Suppose $f : X \to Y$ is a morphism of finite type between noetherian schemes with Y integral. Then there is a dense open subset of Y over which X is flat.*

Exercise 33.9 (Flattening stratification)**.** Under the assumption of the theorem, show that there is a stratification of Y into locally closed subschemes Y_i such that $f^{-1}Y_i$ is flat over Y_i.[2]

Exercise 33.10. Generalize the theorem to a quasicoherent sheaf of finite type on X.

33.3 Fiber dimension

Let $f : X \to Y$ be a morphism of schemes. The *fiber* of f over $y \in Y$ is the scheme $f^{-1}y = y \times_Y X$. We write $\dim_x X = \dim \mathcal{O}_{X,x}$.

Theorem 33.11 ([Har77, Proposition 9.5])**.** *Let $f : X \to Y$ be a flat morphism between locally noetherian schemes. For any $x \in X$ we have*

$$\dim_x X_y + \dim_{f(x)} Y = \dim_x X.$$

Exercise 33.12. (i) Give an example of a non-flat morphism of noetherian schemes where the conclusion of the theorem fails.

Exercise 33.13. Prove the theorem:

(i) Show it is sufficient to assume $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ and both A and B are local rings.

(ii) Pick $t \in A$ not contained in any minimal prime. Show that $\dim A/t = \dim A - 1$.

(iii) With t as above, show that f^*t is not contained in any minimal prime of B. (Hint: Use the fact that the image of f is stable under generization, hence contains all generic points of $\operatorname{Spec} A$.)

(iv) Conclude that $f^{-1}V(t) \subset X$ has dimension $\dim X - 1$.

(v) Use induction on $\dim Y$ to deduce that $\dim X = \dim Y + \dim f^{-1}y$.

Theorem 33.14. *Let $f : X \to Y$ be a morphism of finite type between locally noetherian schemes. Then $\dim X_{f(x)}$ is an upper semicontinuous function of $x \in X$.*

Exercise 33.15. Let $f : X \to Y$ be a proper morphism of locally noetherian schemes. Show that fiber $\dim X_y$ is an upper semicontinuous function of $y \in Y$.

Exercise 33.16. Eliminate the noetherian hypotheses in the second theorem.

33.4 Criteria for flatness

Reading 33.17. [Sta15, Tag 00MD], [Vak14, §24.6]

[2] This is weaker than the usual notion. Generally a flattening stratification is also required to be a *universal flattening* [Sta15, Tag 052F].

The homological criterion

Exercise 33.18. Let M be an A-module.

(i) Show that $M \otimes_A N$ is a right exact functor of N but is not exact in general.

(ii) Suppose that

$$0 \to N' \to N \to N'' \to 0 \tag{33.1}$$

is an exact sequence. Show that

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact if *either* M or N is projective.

(iii) Let N be any A-module and choose a surjection $P_0 \to N$ where P is projective. Let P_1 be the kernel. Define T_1^P to be the kernel of $M \otimes_A P_1 \to M \otimes_A P_0$. Show that T_1^P depends on P only up to canonical isomorphism.

(iv) Write $\mathrm{Tor}_1(M, N)$ for the module constructed above. Show that there is an exact sequence

$$\mathrm{Tor}_1(M, N') \to \mathrm{Tor}_1(M, N) \to \mathrm{Tor}_1(M, N'') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

associated to any exact sequence (33.1).

(v) Prove that $\mathrm{Tor}_1(M, N) = 0$ if either M or N is projective.

(vi) Prove that $\mathrm{Tor}_1(M, N) = \mathrm{Tor}_1(N, M)$.

(vii) Prove that M is flat if and only if $\mathrm{Tor}_1(M, N) = 0$ for all A-modules N if and only if $\mathrm{Tor}_1(N, M) = 0$ for all A-modules N.

Exercise 33.19. (i) Show that an A-module M is flat if and only if for every injection of A-modules $N' \to N$, the induced map

$$M \otimes_A N' \to M \otimes_A N$$

is injective.

(ii) Show that in the previous condition, it is sufficient to assume N' and N are finitely generated.

Exercise 33.20. Let M be a finitely generated A-module. Show M is flat if and only if $I \otimes_A M \to IM$ is a bijection for all ideals $I \subset A$.

The local criterion

Exercise 33.21. Show that M is flat if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of A.

Theorem 33.22 ([Vak14, Theorem 24.6.1])**.** *Suppose that $A \to B$ is a local homomorphism of noetherian local rings and M is a finitely generated B-module. Let k be the residue field of A. Then M is A-flat if and only if* $\mathrm{Tor}_1^A(M, k) = 0$.

It is clear that flatness of M implies $\mathrm{Tor}_1^A(M,k) = 0$. We work on the converse. Assume for the rest of the discussion that $\mathrm{Tor}_1^A(M,k) = 0$.

Exercise 33.23. Show that $\mathrm{Tor}_1^A(M,N) = 0$ if $\mathfrak{m}^n N = 0$ for some positive integer n. (Hint: Reduce to the case where $\mathfrak{m}N = 0$ using the long exact sequence, and then observe that $N \simeq k^{\oplus r}$ as an A-module in that case.)

Exercise 33.24. Use the Artin–Rees lemma to prove the following statements about modules over a noetherian local ring B with maximal ideal $\mathfrak{n}$:

(i) If P is a finitely generated B-module and Q is a submodule then $Q \cap \mathfrak{n}^k P \subset \mathfrak{n}^{k-\ell} Q$ for some ℓ and all $k \gg 0$.

(ii) If P is a finitely generated B-module then $\bigcap \mathfrak{n}^k Q = 0$.

The slicing criterion

Theorem 33.25 ([Vak14, Theorem 24.6.5])**.** *Suppose $A \to B$ is a local homomorphism of local rings and t is not a zero divisor in A. Then B/tB is flat over A/tA if and only if B is flat over A and t is not a zero divisor in B.*

Exercise 33.26. Prove the theorem:

(i) Suppose B is flat over A. Prove that t is not a zero divisor in B if and only if $\mathrm{Tor}_1^A(B, A/tA) = 0$.

(ii) Suppose B is flat over A. Show that $B \otimes_A A'$ is flat over A'. Conclude that B/tA is flat over A/tA.

(iii) Suppose that t is not a zero divisor in B. Show that $\mathrm{Tor}_1^A(k, B) = \mathrm{Tor}_1^{A/tA}(k, B/tB)$.

(iv) Prove the theorem.

The infinitesimal criterion

The equational criterion

33.5 Bézout's theorem

In our proof of Bézout's theorem in Section C, we showed that there was an open subset $U \subset \mathbf{A}^N$ such that $p^{-1}U$ is proper over U. By construction, X is affine over $\mathbf{A}^N$ so $p^{-1}U$ is both proper and affine, hence finite over U.

Exercise 33.27. Show that $p^{-1}U$ is flat over U.

Exercise 33.28. Show that a flat module that is of finite presentation is locally free.

Exercise 33.29. Conclude that $\dim_{\mathbf{k}(q)} \mathcal{O}_{p^{-1}(q)}$ is independent of $q \in U$.

34 Flatness II

35 Flatness III

Chapter 13

Projective space

36 Group schemes and quotients

36.1 The multiplicative group

Reading 36.1. [Vak14, §6.6]

Definition 36.2. A *group scheme* is a scheme G, equipped with the structure of a group on $G(S)$ for every scheme S, such that $G(S) \to G(T)$ is a group homomorphism whenever $T \to S$ is a morphism of schemes.

Exercise 36.3. Define $\mathbf{G}_m(S) = \Gamma(S, \mathcal{O}_S^*)$. Show that $\mathbf{G}_m$ is representable by the scheme $\mathbf{A}^1 \smallsetminus \{0\} = \operatorname{Spec} \mathbf{Z}[t, t^{-1}]$.

Exercise 36.4. (i) Show that the structure of a group scheme on $\operatorname{Spec} A$ induces a homomorphisms of commutative rings:

(a) (comultiplication) $\Delta : A \to A \otimes A$

(b) (antipode) $\iota : A \to A$

(c) (counit) $\epsilon : A \to \mathbf{Z}$

corresponding to multiplication, inversion, and the identity element. (Note that ϵ can also be viewed as a map from A into *every* commutative ring B.)

(ii) Translate the axioms of a group into identities satisfies by these maps. This structure is called a *Hopf algebra*.

Exercise 36.5. Describe the Hopf algebra structure on $\mathbf{Z}[t, t^{-1}]$ corresponding to the group structure on $\mathbf{G}_m$.

Definition 36.6. An action of a group scheme G on a scheme X is a morphism $G \times X \to X$ such that $G(S) \times X(S) \to X(S)$ is an action of $G(S)$ on $X(S)$ for all schemes S.

Exercise 36.7. Show that an action of $\mathbf{G}_m$ on an affine scheme $X = \operatorname{Spec} A$ corresponds to a grading of A by $\mathbf{Z}$.

36.2 Graded rings and quotients

Let $X = \operatorname{Spec} A$ be an affine scheme with an action of $\mathbf{G}_m$. This corresponds to a grading of A by $\mathbf{Z}$, as we saw in the last section.

Definition 36.8. Let G be an algebraic group acting on a scheme X. The *fixed locus* of X is the functor $X^G \subset X$ consisting of all $x \in X$ such that $g.x = x$ for all $g \in G$. More precisely, $X^G(S)$ is the set of all $x \in X(S)$ such that for all S-schemes T and all $g \in G(T)$ we have $g.x\big|_T = x\big|_T$.

Exercise 36.9. Let $\mathbf{G}_m$ act on an affine scheme $X = \operatorname{Spec} A$. Show that the fixed locus is $V(A_+)$ where A_+ is the ideal generated by elements of nonzero degree.

Definition 36.10. Let X be a scheme with an action of an algebraic group G. If it exists, the initial G-morphism from X to a scheme on which G acts trivially is called the *quotient* of X by G. It is denoted X/G if it exists.

Exercise 36.11. Let $\mathbf{G}_m$ act on $X = \operatorname{Spec} A$. Show that the $D(f)$, as f ranges among homogeneous elements of A, form a basis for the $\mathbf{G}_m$-invariants open subsets of X.

Exercise 36.12. Show that when an algebraic group G acts on $X \times G$ by $g.(x, h) = (x, gh)$, the quotient $(X \times G)/G$ is X.

Theorem 36.13 ([MFK, Chapter 1, Theorem 1.1])**.** *Suppose that $\mathbf{G}_m$ acts on an affine scheme $X = \operatorname{Spec} A$, corresponding to a grading $A = \sum A_n$. Show that $X/\mathbf{G}_m$ exists and is equal to* $\operatorname{Spec} A_0$.[1]

Exercise 36.14. Let $X = \operatorname{Spec} A$ be an affine scheme with an action of $\mathbf{G}_m$ corresponding to a grading $A = \sum A_n$. Let $X^\circ \subset X$ be the complement of $X^G \subset X$. Show that $X^\circ/\mathbf{G}_m$ exists and is equal to $\operatorname{Proj} A$:

(i) Show that $\mathbf{G}_m$ acts on X°.

(ii) Show that $D(f)$, for $f \in A$ homogeneous of nonzero degree, form a basis for the $\mathbf{G}_m$-invariant open subsets of X°.

(iii) Show that for each $f \in A_+$, the quotient $D(f)/\mathbf{G}_m$ exists and is equal to $\operatorname{Spec} A[f^{-1}]_0 = \operatorname{Proj} A[f^{-1}]$.

(iv) Construct a map $X^\circ \to \operatorname{Proj} A$ and show that it has the universal property of $X/\mathbf{G}_m$.

37 Quasicoherent sheaves and graded modules

Reading 37.1. [Har77, §II.5]

Definition 37.2. Let A be a graded ring. A *graded A-module* is an A-module M that is decomposed as a direct sum $M = \sum M_n$ with $A_m M_n \subset M_{m+n}$ for all $m, n \in \mathbf{Z}$.

This is essentially equivalent to Exercise 37.5. You might want to regard this exercise as a hint for or a step in the solution of that one.

Exercise 37.3. Let A be a graded ring, corresponding to a comultiplication map $\mu^* : A \to A[t, t^{-1}]$. Show that to give a grading on an A-module M is the same as to give a map $\mu^* : M \to M[t, t^{-1}]$ such that $\mu^*(fx) = \mu^*(f)\mu^*(x)$ for any $f \in A$ and $x \in M$. Interpret this geometrically as an isomorphism $\mu^* \widetilde{M} \simeq p^* \widetilde{M}$ where $p : \mathbf{G}_m \times \operatorname{Spec} A \to \operatorname{Spec} A$ and $\mu : \mathbf{G}_m \times \operatorname{Spec} A \to \operatorname{Spec} A$ are, respectively, the projection and the action.

[1] This holds more generally for an action of a reductive group scheme.

Exercise 37.4 (Flat base change for global sections)**.** Consider a cartesian diagram of schemes:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Assume that p is coherent (quasicompact and quasiseparated). Show that $g^* p_* \mathscr{F} = p'_* f^* \mathscr{F}$ for any quasicoherent sheaf $\mathscr{F}$ on Y'.

Exercise 37.5. Let $X = \operatorname{Spec} A$ and let $Y = \operatorname{Proj} A$. Write $\pi : X^\circ \to Y$ for the projection and let $j : X^\circ \to X$ be the inclusion. Suppose that $\mathscr{F}$ is a quasicoherent sheaf on Y. Show that $\pi^* \mathscr{F}$.

(i) Show that $j_* \pi^* \mathscr{F}$ is a quasicoherent sheaf on X. Conclude that $j_* \pi^* \mathscr{F} = \widetilde{M}$ for some A-module M.

(ii) Show that M is naturally equipped with the structure of a graded A-module. (Hint: Pull back via the projection $p : \mathbf{G}_m \times X \to X$ and $\mu : \mathbf{G}_m \times X \to X$ and compare.)

Exercise 37.6. Suppose that A is a graded ring and M is a graded A-module. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Proj} A$, and let $\pi : X^\circ \to Y$ be the projection. Let $\mathscr{F}$ be the sheaf on X associated to M. Define $\mathscr{G}(U) = \mathscr{F}(\pi^{-1} U)_0$ for all open $U \subset Y$.

(i) Show that $\mathscr{G}$ is a sheaf on Y.

(ii) Suppose that $\mathscr{F} = j_* \pi^* \mathscr{F}'$ for a quasicoherent sheaf on Y. Construct a canonical isomorphism $\mathscr{G} \simeq \mathscr{F}'$.

Exercise 37.7. Let $X = \operatorname{Spec} A$ and let $Y = \operatorname{Proj} A$. Assume that A_+ is generated by elements of degrees 1 and -1.

(i) Show that the category $\mathbf{QCoh}(Y)$ is equivalent to the category of graded quasicoherent sheaves of $\mathcal{O}_{X^\circ}$-modules.

(ii) Show that the category $\mathbf{QCoh}(X^\circ)$ is equivalent to the category of objects $\mathscr{F} \in \mathbf{QCoh}(X)$ such that $\mathscr{F} \to j_* j^* \mathscr{F}$ is an isomorphism.

38 Line bundles and divisors

Reading 38.1. [GD67, IV.21], [Har77, II.6], [Vak14, Chapter 14]

Invertible sheaves are examples of quasicoherent sheaves, so we can use the classification of quasicoherent sheaves on projective space to classify line bundles.

Exercise 38.2. Suppose that $\mathscr{L}$ and $\mathscr{L}'$ are invertible sheaves. Show that $\mathscr{L} \otimes \mathscr{L}'$ and $\underline{\mathrm{Hom}}(\mathscr{L}, \mathscr{L}')$ are invertible sheaves as well.[2] Show that isomorphism classes of invertible sheaves on a scheme X form an abelian group where addition is $\otimes$, difference is $\underline{\mathrm{Hom}}$, and the zero element is $\mathcal{O}_X$.

[2] The notation $\underline{\mathrm{Hom}}$ refers to the sheaf of homomorphisms.

Exercise 38.3. Let $A = \mathbf{Z}[x_0, \ldots, x_n]$ and $X = \operatorname{Spec} A$ and $Y = \operatorname{Proj} A$. Construct an equivalence of categories between the category of line bundles on Y and the category of graded invertible sheaves on X°.

We will have fully classified invertible sheaves on $\mathbf{P}^n$ when we show that

(1) a sheaf on X° is invertible if and only if $j_* X^\circ$ is invertible, and

(2) all invertible sheaves on $X = \mathbf{A}^{n+1}$ are trivial.

38.1 Cartier divisors

Definition 38.4 (Meromorphic functions)**.** Let X be a scheme. Let $\mathscr{M}_X$ be the sheaf obtained by adjoining inverses to all nondivisors of zero in $\mathcal{O}_X$. This is known as the sheaf of *meromorphic functions* on X. An invertible sheaf on X is called an *invertible fractional ideal* if it can be embedded, as an $\mathcal{O}_X$-module, in $\mathscr{M}_X$.

Exercise 38.5. Show that there is an injection $\mathcal{O}_X \to \mathscr{M}_X^*$.

Definition 38.6 (Cartier divisors)**.** Let $\underline{\mathrm{Div}}_X = \mathscr{M}_X^* / \mathcal{O}_X^*$. This is known as the sheaf of *Cartier divisors* on X. If f is a section of $\mathscr{M}_X$, the associated divisor is denoted (f). Divisors associated to meromorphic functions are called *principal*.

Exercise 38.7. Suppose that $X = \operatorname{Spec} A$ and A is a unique factorization domain. Show that the map

$$\Gamma(X, \mathscr{M}_X^*) \to \Gamma(X, \underline{\mathrm{Div}}_X)$$

is a surjection.

Exercise 38.8. Suppose that X is an integral scheme with generic point η. Show that $\mathscr{M}_X(U) = \mathbf{k}(\eta)$ for all nonempty $U \subset X$.

Exercise 38.9. Let D be a divisor on X (an element of $\Gamma(X, \underline{\mathrm{Div}}_X)$).

Exercise 38.10. Suppose that D and E are Cartier divisors. We say that $D \geq E$ if $D - E = (f)$ for some $f \in \mathcal{O}_X$. Show that this gives $\underline{\mathrm{Div}}_X$ the structure of a sheaf of partially ordered groups. Let $\underline{\mathrm{Div}}_X^+$ be the subsheaf of divisors $D \in \underline{\mathrm{Div}}_X$ such that $D \geq 0$.

Exercise 38.11. Let D be a divisor on X. Let $\mathcal{O}_X(D)$ be the set of $f \in \mathscr{M}_X$ such that $(f) \geq -D$.

(i) Show that $\mathcal{O}_X(D)$ is an invertible sheaf on X.

(ii) Show that this gives a map

$$\Gamma(X, \underline{\mathrm{Div}}_X) \to \operatorname{Pic}(X).$$

(iii) Show that the image of this map consists of all equivalence classes of invertible fractional ideals of X.

Exercise 38.12. Suppose that X is an integral scheme. Show that every invertible sheaf is isomorphic to an invertible fractional ideal. Conclude that there is an isomorphism:

$$\Gamma(X, \underline{\mathrm{Div}}_X) / \Gamma(X, \mathscr{M}_X^*) \simeq \operatorname{Pic}(X)$$

38.2 Weil divisors

Let X be a locally noetherian scheme and $x \in X$ a point. We say that x has codimension 1 in X if $\dim \mathcal{O}_{X,x} = 1$.

Definition 38.13 (Weil divisor)**.** Suppose X is noetherian. A *Weil divisor* on X is a formal sum of codimension 1 points of X. The abelian group of Weil divisors is denoted $Z^1(X)$.

Let $D \geq 0$ be a Cartier divisor on X. Then $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ so it is an ideal. It therefore defines a closed subscheme $V(\mathcal{O}_X(-D))$. Furthermore, it defines a Weil divisor: For each codimension 1 point $x \in X$, set

$$c_x(D) = \operatorname{length} \mathcal{O}_{X,x}/\mathcal{O}_{X,x}(-D).$$

Exercise 38.14. Show that $c_x(D) = 0$ for all but finitely many points $x \in X$. Conclude that $c(D) = \sum c_x(D)[x]$ is a Weil divisor of X.

Exercise 38.15. Show that $c_x(D + E) = c_x(D) + c_x(E)$. Conclude that c_x extends to homomorphisms defined on $\operatorname{Div}(X) \to \mathbf{Z}$ and c extends to $\operatorname{Div}(X) \to Z^1(X)$.

Exercise 38.16 ([GD67, Théorème (IV.21.6.9)])**.** An element of $Z^1(X)$ is called locally principal if is locally $c([f])$ for some $f \in \mathscr{M}_X$. Show that $\operatorname{Div}(X) \to Z^1(X)$ is injective and its image consists of the locally principal cycles.

38.3 The Picard group of projective space

Exercise 38.17. Let A be a graded ring and let $X = \operatorname{Spec} A$ and $Y = \operatorname{Proj} A$. If M is an A-module, define $M(n)_k = M(n + k)$.

(i) Show that $A(n)$ is an invertible sheaf on $\operatorname{Proj} A$ for all $n \in \mathbf{Z}$. This sheaf is denoted $\mathcal{O}_Y(n)$. We write $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n)$.

(ii) Show that $\Gamma(Y, \mathscr{F}(n)) = M_n$ whenever $\widetilde{M} = j_*\pi^*\mathscr{F}$.

Exercise 38.18. Let $A = \mathbf{Z}[x_0, \ldots, x_n]$ and let $X = \operatorname{Spec} A = \mathbf{A}^{n+1}$ and $Y = \operatorname{Proj} A = \mathbf{P}^n$. Show that a quasicoherent sheaf $\mathscr{L}$ on Y is invertible if and only if $\pi^*\mathscr{L}$ is invertible if and only if $j_*\pi^*\mathscr{L}$ is invertible.[3]

Exercise 38.19. Prove that $\operatorname{Pic} \mathbf{A}^n = 0$.

Exercise 38.20. Prove directly that $\operatorname{Pic}(\mathbf{A}^n \smallsetminus \{0\}) = 0$. This gives another solution to Exercise 38.18.

Exercise 38.21. Prove that $\operatorname{Pic} \mathbf{P}^n = \mathbf{Z}$ with $1 \in \mathbf{Z}$ corresponding to $\mathcal{O}(1)$.

[3] This calculation also works over a field. With small modification, it even works over an arbitrary base ring replacing $\mathbf{Z}$.

Part III

Cohomology

Chapter 14

Sheaf cohomology

39 Divisors

40 Sheaves III

40.1 Injective resolutions

Definition 40.1. Let X be a scheme.[1] A sheaf of $\mathcal{O}_X$-modules $\mathscr{I}$ is said to be *injective* if $\operatorname{Hom}_{\mathcal{O}_X\text{-}\mathbf{Mod}}(\mathscr{F}, \mathscr{I})$ is an exact functor of $\mathscr{F}$.

Should be easy, but it's important to know and understand why this is true

Exercise 40.2. Show that $\mathscr{I}$ is injective if and only if, for every injection of sheaves of $\mathcal{O}_X$-modules $\mathscr{F}' \to \mathscr{F}$, the map $\operatorname{Hom}(\mathscr{F}, \mathscr{I}) \to \operatorname{Hom}(\mathscr{F}', \mathscr{I})$ is surjective.

Theorem 40.3 (Grothendieck [Gro57, Théorème 1.10.1])**.** *Every sheaf of $\mathcal{O}_X$-modules can be embedded in an injective module.*

The proof uses a few facts about the category of $\mathcal{O}_X$-modules:

(i) the category is *abelian*: it has kernels, cokernels, and images that behave as we are accustomed;

(ii) the category has a *set of generators*: every object is a quotient of a direct sum of $\mathcal{O}_U$ for $U \subset X$ open;

(iii) arbitrary (small) colimits exist and filtered colimits are exact.

The proof is known as the 'small object argument'. The idea is that if we have a witness $\mathscr{F}' \subset \mathscr{F}$ to the failure of injectivity of a sheaf $\mathscr{I}$ then we pushout:

$$\begin{tikzcd} \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \\ \mathscr{I} & \end{tikzcd}$$

Iterating this process enough, we get an injective module. The details of the proof will be a series of exercises, following [Gro57, §1.10].

[1] or, really, a ringed space

Exercise 40.4. Show that the category of sheaves of $\mathcal{O}_X$-modules has a generator.[2] (Hint: Take the direct sum of all $\mathcal{O}_U$, with U ranging among open subsets of X.)

Exercise 40.5. Let $\mathscr{G}$ be the generator. Show that $\mathscr{I}$ is injective if and only if, for every subobject $\mathscr{G}' \subset \mathscr{G}$, every morphism $\mathscr{G}' \to \mathscr{I}$ extends to $\mathscr{G} \to \mathscr{I}$.

Exercise 40.6. Fix $\mathscr{F}$. For each successor ordinal $n+1$, let $\mathscr{F}_{n+1}$ be the pushout of the diagram below:

$$\begin{tikzcd} \sum_{\mathscr{G}' \subset \mathscr{G}} \mathscr{G}' \times \mathrm{Hom}(\mathscr{G}', \mathscr{F}_n) \arrow[r] \arrow[d] & \sum_{\mathscr{G}' \subset \mathscr{G}} \mathscr{G} \times \mathrm{Hom}(\mathscr{G}', \mathscr{G}_n) \\ \mathscr{F}_n & \end{tikzcd}$$

When n is a limit ordinal, let $\mathscr{F}_n = \varinjlim_{m<n} \mathscr{F}_m$. Show that $\mathscr{F}_n$ is injective for large n.

40.2 Flaccid sheaves

Definition 40.7. A sheaf $\mathscr{I}$ is said to be *flaccid* if

$$\mathscr{I}(U) \to \mathscr{I}(V)$$

is surjective for all open $V \subset U$.

Exercise 40.8. Show that injective sheaves are flaccid.

Exercise 40.9. Suppose that

$$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$$

is exact and $\mathscr{A}$ is flaccid.

(i) Show that

$$0 \to \Gamma(X, \mathscr{A}) \to \Gamma(X, \mathscr{B}) \to \Gamma(X, \mathscr{C}) \to 0$$

is exact.

(ii) Show that $\mathscr{B}$ is flaccid if and only if $\mathscr{C}$ is flaccid.

40.3 Cohomology as a derived functor

Definition 40.10. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. Choose an embedding $\mathscr{F} \subset \mathscr{I}$ where $\mathscr{I}$ is a flaccid $\mathcal{O}_X$-module. Define

$$\begin{aligned} H^1(X, \mathscr{F}) &= \Gamma(X, \mathscr{I}/\mathscr{F})/\Gamma(X, \mathscr{I}) \\ H^n(X, \mathscr{F}) &= H^{n-1}(X, \mathscr{I}/\mathscr{F}) \qquad \text{for } n \geq 2. \end{aligned}$$

[2] A generator is an object A such that an injection $B \to C$ is an isomorphism if and only if the induced map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ is an isomorphism.

40.4 Torsors

Definition 40.11 (Torsor)**.** Let $\mathscr{G}$ be a sheaf of groups on X. A *$\mathscr{G}$-torsor* is a sheaf of sets $\mathscr{P}$, equipped with an action of $\mathscr{G}$, such that there is a cover of X by open subsets U such that $\mathscr{P}\big|_U$ is isomorphic to $\mathscr{G}\big|_U$ as a sheaf on U with $\mathscr{G}\big|_U$-action.

A $\mathscr{G}$-torsor is said to be *trivial* if it is isomorphic to $\mathscr{G}$ as a sheaf of $\mathscr{G}$-sets. Define $H^1(X,\mathscr{G})$ to be the set of isomorphism classes of $\mathscr{G}$-torsors on X.

Exercise 40.12. Show that a $\mathscr{G}$-torsor $\mathscr{P}$ is trivial if and only if $\Gamma(X,\mathscr{P}) \neq \varnothing$.

Exercise 40.13. Suppose that $\mathscr{G}$ is a flaccid sheaf of groups. Show that every $\mathscr{G}$-torsor is trivial.

Exercise 40.14. Suppose that $\mathscr{G}$ acts on a sheaf of sets $\mathscr{P}$.

(i) Show that there is an exact sequence

$$0 \to \Gamma(X,\mathscr{G}) \to \Gamma(X,\mathscr{P}) \to \Gamma(X,\mathscr{P}/\mathscr{G}) \to H^1(X,\mathscr{G}).$$

(Hint: Given a section $\sigma \in \Gamma(X,\mathscr{P}/\mathscr{G})$, consider its preimage in $\mathscr{P}$.)

(ii) Show that the last arrow is surjective if $\mathscr{P}$ is flaccid.

Exercise 40.15. Prove that our two definitions of $H^1(X,\mathscr{G})$ coincide when $\mathscr{G}$ is a sheaf of abelian groups.

Exercise 40.16. Show that the sheaf of isomorphisms between two $\mathscr{G}$-torsors is naturally equipped with the structure of a $\mathscr{G}$-torsor by $(g.f)(x) = g.(f(x))$. Show that $[\mathscr{Q}] - [\mathscr{P}] = [\underline{\mathrm{Hom}}(\mathscr{P},\mathscr{Q})]$ in the additive structure of $H^1(X,\mathscr{G})$ induced from the derived functor construction.

40.5 Line bundles

Exercise 40.17. Construct an equivalence of categories between the category of line bundles on a scheme X and the category of torsors on X under the group $\mathbf{G}_m$.

41 Čech cohomology

Exercise 41.1. Let $\mathscr{F}$ be a sheaf of abelian groups on X. For each open $U \subset X$, define $\mathscr{H}^p\mathscr{F}(U) = H^p(U,\mathscr{F})$.

(i) Show that $\mathscr{H}^p\mathscr{F}$ is naturally a presheaf on X. (Hint: The restriction of a flaccid sheaf to an open subset is still flaccid, and restriction of sheaves is exact.)

(ii) Show that the sheafification of $\mathscr{H}^p\mathscr{F}$ is the zero sheaf for all $p > 0$.

Definition 41.2. For each n, let $\mathfrak{U}_n$ be the set of all symbols $U_1 \wedge \cdots \wedge U_n$. Define $C^p(\mathfrak{U},\mathscr{F})$ to be the set of functions σ on $\mathfrak{U}_n$ with

$$\sigma(U_1 \wedge \cdots \wedge U_n) \in \mathscr{F}(U_1 \cap \cdots \cap U_n)$$
$$\sigma(U_{f(1)} \wedge \cdots \wedge U_{f(n)}) = \mathrm{sgn}(f)\sigma(U_1 \wedge \cdots \wedge U_n).$$

These are the *Čech p-cochains*. Define a *coboundary map*

$$C^p(\mathfrak{U}, \mathscr{F}) \to C^{p+1}(\mathfrak{U}, \mathscr{F})$$

by defining $d(\sigma)(U_1 \wedge \cdots \wedge U_{p+1}) = \sum (-1)^i \sigma_{U_1,\ldots,\hat{U}_i,\ldots,U_{p+1}}\big|_{U_1 \cap \cdots \cap U_{p+1}}$. The cohomology of this complex is called the *Čech cohomology* of $\mathscr{F}$ with respect to $\mathfrak{U}$ and is denoted $H^*(\mathfrak{U}, \mathscr{F})$.[3]

Exercise 41.3. (i) Construct a map $H^1(\mathfrak{U}, \mathscr{G}) \to H^1(X, \mathscr{G})$.

(ii) Show that the image consists of all $\mathscr{G}$-torsors $\mathscr{P}$ such that $\mathscr{P}\big|_U$ trivial for all U in $\mathfrak{U}$.

(iii) Conclude that $\check{H}^1(X, \mathscr{G}) = H^1(X, \mathscr{G})$ for all sheaves of groups $\mathscr{G}$ on X.

Exercise 41.4. Suppose that $\mathscr{I}$ is a flaccid sheaf. Show that $H^p(\mathfrak{U}, \mathscr{I}) = 0$ for all $p > 0$. (Hint: Realize the Čech complex as the global sections of an exact sequence of sheaves.) Conclude that Čech cohomology agrees with sheaf cohomology for flaccid sheaves.

41.1 Affine schemes

Exercise 41.5. Prove $H^1(X, \mathscr{G}) = 0$ for all quasicoherent sheaves $\mathscr{G}$ on all affine scheme X.

Soon we will see that $H^n(X, \mathscr{G}) = 0$ for $n > 0$ and all quasicoherent sheaves $\mathscr{G}$ on all affine schemes X.

Exercise 41.6. Prove that the Čech complex is exact for any quasicoherent sheaf on an affine scheme and any cover by distinguished open affines. Conclude that $\check{H}^n(X, \mathscr{F}) = 0$ for all $n > 0$ when X is affine and $\mathscr{F}$ is quasicoherent.

[3]The *Čech cohomology* is defined by taking a colimit of $H^*(\mathfrak{U}, \mathscr{F})$ over all covers, ordered by refinement. In order for this to make sense and be a filtered colimit, it is best to define a cover to be a choice of open neighborhood of each point.

Chapter 15

Lines on a cubic surface

42 Čech cohomology II

42.1 The Čech spectral sequence

Should be simple

Exercise 42.1. Suppose $\mathscr{I}$ is a flaccid sheaf on X. Show that $\mathscr{I}\big|_U$ is also flaccid, for all open $U \subset X$.

Let $\mathscr{F}$ be a sheaf on X and let $\mathscr{I}^\bullet$ be a flaccid resolution of $\mathscr{F}$. Fix a cover $\mathfrak{U}$ of X and write $C^\bullet(\mathfrak{U}, \mathscr{F})$ for the Čech complex of $\mathscr{F}$. We can form a *double complex*:

$$C^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)$$

If we compute the cohomology first with respect to $C^\bullet$, we get $\Gamma(X, \mathscr{I}^\bullet)$, whose cohomology is $H^q(X, \mathscr{F})$. If we compute it first with respect to the $\mathscr{I}^\bullet$ differential, we get a complex whose (p,q)-entry is

$$\prod_{U_1,\ldots,U_p \in \mathfrak{U}} H^q(U_1 \cap \cdots \cap U_p, \mathscr{F}\big|_{U_1 \cap \cdots \cap U_p}).$$

In particular, we find the Čech cohomology as the $q = 0$ column. Suppose that $H^q(U_1 \cap \cdots \cap U_p, \mathscr{F}) = 0$ for all p and all $q > 0$. Then everything vanishes but the Čech cohomology with respect to $\mathfrak{U}$.

Exercise 42.2. Suppose that E is a double complex in the first quadrant. Assume that for all $p > 0$ we have $H^{p,\bullet}E = 0$ and that for all $q > 0$ we have $H^{\bullet,q}E = 0$. (In other words, the columns and rows are all exact, except in degree 0.) Conclude that $H^p H^{0,\bullet}E = H^p H^{\bullet,0}E$ (in a natural way) for all p.

Theorem 42.3 ([Gro57, Théorème 3.8.1])**.** *Suppose $\mathfrak{U}$ is a cover of X and $H^q(U_1 \cap \cdots \cap U_p, \mathscr{F}) = 0$ for all $p, q > 0$ and all $U_1, \ldots, U_p \in \mathfrak{U}$. Then Čech cohomology of $\mathscr{F}$ agrees with derived functor cohomology.*

A more careful analysis of the proof of the exercise above gives a refinement of this theorem. Observe that to get an isomorphism

$$H^p(\mathfrak{U}, \mathscr{F}) \simeq H^p(X, \mathscr{F})$$

we needed the following vanishing:

$$\begin{gathered}H^1(U_1 \cap \cdots \cap U_p, \mathscr{F}) = 0 \\ H^1(U_1 \cap \cdots \cap U_{p-1}, \mathscr{F}) = H^2(U_1 \cap \cdots \cap U_{p-1}, \mathscr{F}) = 0 \\ H^2(U_1 \cap \cdots \cap U_{p-2}, \mathscr{F}) = H^3(U_1 \cap \cdots \cap U_{p-2}, \mathscr{F}) = 0 \\ \vdots\end{gathered}$$

for all $U_1, \ldots, U_k \in \mathfrak{U}$. In particular, we can make the following conclusion:

Theorem 42.4. *If $H^i(U_1 \cap \cdots \cap U_q, \mathscr{F}) = 0$ for $i + q + 1 \le p$ and $i > 0$ then*

$$H^p(\mathfrak{U}, \mathscr{F}) = H^p(X, \mathscr{F}).$$

Corollary 42.4.1. *The cohomology of a quasicoherent sheaf on an affine scheme is trivial in positive degrees.*

Corollary 42.4.2. *The Čech cohomology of a quasicoherent sheaf with respect to an affine cover of a separated scheme agrees with the derived functor cohomology.*

42.2 Cohomology and dimension

Theorem 42.5 ([Gro57, Théorme 3.6.5], [Har77, Theorem III.2.7])**.** *Let X be a noetherian topological space of dimension n. Show that $H^p(X, \mathscr{F}) = 0$ for all sheaves of abelian groups $\mathscr{F}$ on X and all $p > 0$.*

Exercise 42.6. Assume the result holds for all closed subsets of X other than X itself.

(i) Show that the result holds when X is irreducible and $\mathscr{F} = \mathbf{Z}_U$ for some open $U \subset X$.

(ii) Show that the result holds when X is irreducible and $\mathscr{F}$ is a quotient of $\mathbf{Z}_U$ for some open $U \subset X$.

(iii) Show that the result holds when X is irreducible and $\mathscr{F}$ is finitely generated.

(iv) Show that the result holds when X is irreducible.

(v) Show that the result holds for all X.

42.3 Cohomology of invertible sheaves on projective space

Exercise 42.7. Compute $H^p(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$ for all p.

Exercise 42.8. Compute $H^p(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n))$ for all integers n and p. (Hint: There is a map $\mathcal{O}(n) \to \mathcal{O}(n+1)$ by 'multiplication by x'.)

Exercise 42.9. Repeat the above calculation for $\mathbf{P}^2$ and then for all $\mathbf{P}^n$. (Hint: There should be an induction on n going on here. You'll have to compute the cohomology of $\mathcal{O}_{\mathbf{P}^n}$ by hand, though.)

Recall that quasicoherent sheaves on $\mathbf{P}^n$ are equivalent to graded modules on $\mathbf{A}^{n+1}$ such that if $j : \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{A}^{n+1}$ is the inclusion then we have $\mathscr{F} \to j_* j^* \mathscr{F}$ is an isomorphism. We compute the derived functors of this inclusion. Under this equivalence, global sections on $\mathbf{P}^n$ correspond to $\Gamma(\mathbf{A}^{n+1}, \mathscr{F})_0$. Regarding $\mathscr{F} = \widetilde{M}$ for some graded module M, we form the following resolution:

$$0 \to M \to \prod_{1 \le i \le n} M_{x_i} \to \prod_{1 \le i < j \le n} M_{x_i x_j} \to \cdots \to M_{x_0 \cdots x_n} \to 0.$$

43 Čech cohomology III

44 Lines on a cubic surface

For any scheme S, let $G(S)$ be the set of lines in $\mathbf{P}^3$. Equivalently, $G(S)$ is the set of equivalence classes of 2-dimension vector subspaces $V \subset \mathbf{A}^4_S$. This is the Grassmannian.

Exercise 44.1. Show that $\mathbf{Grass}(k, n)$ is proper. (Hint: Use the valuative criterion. For the existence part, let R be a valuation ring with field of fractions K. Represent an element of $\mathbf{Grass}(k, n)(K)$ by an $k \times n$ matrix with entries in R such that not all $k \times k$ minors are zero. Multiply by the inverse of the $k \times k$-submatrix whose determinant has minimal valuation. Argue that the result has entries in R.)

Suppose that $V \subset \mathbf{A}^4_S$ represents a line. This gives an embedding

$$\mathbf{P}_S V \to \mathbf{P}^4_S$$

where $\mathbf{P}_S V$ is the space of lines in the rank 2 vector bundle V over S. Indeed, any 1-dimensional subspace of $\mathbf{P}_S V$ is also a 1-dimensional subspace of $\mathbf{A}^4_S$. If we choose an isomorphism $V \simeq \mathbf{A}^2_S$ then we get an isomorphism $\mathbf{P}_S V \simeq \mathbf{P}^1_S$.

This construction has an inverse: If we have a closed embedding of S-schemes $f : P \subset \mathbf{P}^3_S$ then it is given by a tuple $(\mathscr{L}, x_0, \ldots, x_3)$ with the x_i generating $\mathscr{L}$. Let $\pi : P \to S$ be the projection. Then $x_0, \ldots, x_3$ give a map $\mathcal{O}^4_S \to \pi_* \mathscr{L}$. If P is isomorphic to $\mathbf{P}^1_S$ locally in S and $\mathscr{L}$ is locally isomorphic to $\mathcal{O}_{\mathbf{P}^1_S}(1)$ then $\pi_* \mathscr{L}$ is a locally free sheaf of rank 2 on S. If we show the map $\mathcal{O}^4_S \to \pi_* \mathscr{L}$ is surjective then $\mathbf{V}(\pi_* \mathscr{L}) \to \mathbf{V}(\mathcal{O}^4_S) \simeq \mathbf{P}^3_S$ will be a closed embedding.

To see that $\mathcal{O}^4_S \to \pi_* \mathscr{L}$ is a surjection, it is sufficient to treat the case when S is a point. (A morphism of finite rank vector bundles that is a surjection fiberwise is a surjection, by Nakayama's lemma.) Now this corresponds to a degree 1 morphism of graded modules $k[x, y] \to k[x, y]$ that is surjective modulo x and modulo y. It follows that the image contains both x and y so it is surjective on global sections.

Exercise 44.2. Let $H(S)$ be the set of closed embeddings of S-schemes $f : P \subset \mathbf{P}^n_S$ such that locally in S, we have $P \simeq \mathbf{P}^k_S$ and $f^* \mathcal{O}_{\mathbf{P}^n_S}(1) = \mathcal{O}_{\mathbf{P}^k_S}(1)$. Prove that $H \simeq \mathbf{Grass}(k + 1, n + 1)$.

Let $Y = \mathbf{A}^N$ be the space of all homogeneous cubics in 4 variables. Let X be the functor with $X(S)$ equal to the set of pairs (p, L) where p is a homogeneous cubic in 4 variables, and L is family of lines in $\mathbf{P}^3$ parameterized by S such that L lies on the cubic surface defined by p. If we view p as a morphism $\mathbf{A}^4_S \to \mathbf{A}^1_S$ and L as a 2-dimensional linear subspace $V \subset \mathbf{A}^4_S$ then the condition defining X is $p(V) = 0$.

Exercise 44.3. (i) Show that X is representable by a scheme.

(ii) Show that X is proper over Y. (Hint: Make use of the properness of the Grassmannian.)

Exercise 44.4. Give an example of a non-smooth cubic surface. Show that the number of lines on a non-smooth cubic surface does not have to be the same as the number of lines on a smooth cubic surface.

Let $Z(S)$ be the set of pairs (p, x) where p is a homogeneous cubic and x is a point of the cubic surface defined by p. In other words, $Z(S)$ consists of $p : \mathbf{A}^4_S \to \mathbf{A}^1_S$ and x is represented by a 1-dimensional subspace $W \subset \mathbf{A}^4_S$ with $p(W) = 0$. Let $Z_0 \subset Z$ be the set of all points (p, x) such that Z is not smooth over Y at x.

Exercise 44.5. (i) Show that Z fails to be smooth at (p, x) if and only if $p'(x) = 0$. (You might need to work locally in Z to make sense of this condition.)

(ii) Conclude that $Z_0 \subset Z$ is closed.

(iii) Conclude that the $p \in Y$ defining smooth cubic surfaces form an open subset, denoted $Y^\circ \subset Y$.

Let X° be the preimage of Y°. Points of X° correspond to lines on smooth cubic surfaces.

Exercise 44.6. Let X' be a cubic surface over S' and $L \subset X$ a line. Suppose that $S \subset S'$ is a square-zero extension with ideal J and $\pi : X \to S$ is the restriction of X' to S. Let $\tau : L \to S$ be the restriction of π. Show that there is an obstruction to extending L to a line $L' \subset X'$ over S' lying in $\mathrm{Ext}^1_{\mathcal{O}_L}(\mathscr{I}/\mathscr{I}^2, \pi^* J)$ and that extensions are parameterized by $\mathrm{Hom}_{\mathcal{O}_L}(\mathscr{I}/\mathscr{I}^2, \tau^* J)$.

Exercise 44.7. With notation as in the last exercise, show that $\mathscr{I}/\mathscr{I}^2 = \mathcal{O}_L(1)$.

Exercise 44.8. Prove that $\mathrm{Ext}^i(\mathscr{I}/\mathscr{I}^2, \tau^* J) = 0$ for $i = 0, 1$. (Suggestion: The case where $S = \operatorname{Spec} k$ and $J = k$ is all we really need, so feel free to do just that.)

Part IV

Deleted material

Chapter 16

Deformation theory

45 Formal functions

46 Locally trivial deformation problems

46.1 Deforming morphisms to a smooth scheme

46.2 Deforming smooth schemes

46.3 Deforming vector bundles

47 Homogeneous functors and Schlessinger's criteria

D Lines on a cubic surface

48 Divisors and line bundles

49 Associated points and the field of fractions

Reading 49.1. [Vak14, §5.5], [Eis91, §3.1]

Every integral domain can be embedded in a field, but not every commutative ring can. We will see that there is a replacement for the field of fractions, called the total ring of fractions, obtained by localizing the ring at its associated primes, or equivalently by inverting all nondivisors of zero. The only associated prime of an integral domain is the zero ideal, so the total ring of fractions recovers the field of fractions in this case.

Definition 49.2. Let A be a commutative ring and M an A-module. For any subset $S \subset M$, the *annihilator*[1] of S in A is the set of all $a \in A$ such that $ax = 0$. It is denoted $\mathrm{Ann}(S)$.

[1] The French use the more evocative *assassin*.

Definition 49.3. Let A be a noetherian ring and let M be a finitely generated A-module.[2] A prime $\mathfrak{p}$ of A is said to be *associated to* M if there is an injection of A-modules $A/\mathfrak{p} \to M$. The set of primes assocated to M is denoted $\mathrm{Ass}(M)$.

Should be quick and easy and not worth writing down.

Exercise 49.4. Show that $\mathfrak{p}$ is associated to M if and only if there is some $x \in M$ such that $\mathrm{Ann}_A(x) = \mathfrak{p}$.

Exercise 49.5 ([Vak14, Exercise 5.5.J], [Eis91, Proposition 3.4])**.** Consider the collection of all proper ideals of A that are annihilators of elements of M, ordered by inclusion. Show that the maximal elements of this collection are associated primes of M.

Exercise 49.6 ([Vak14, Exercise 5.5.K])**.** Show that $M \to \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}$ is injective.

Exercise 49.7. Show that the union of the associated primes of a noetherian commutative ring A is the set of zero divisors of A.[3]

Exercise 49.8. Let A be a commutative noetherian ring, M an A-module, and $f \in A$. Show that $\mathrm{Ass}_{A_f} M_f = D(f) \cap \mathrm{Ass}_A M$.

49.1 Normalization

49.2 Intuition from topology

Let k be a field. Fix a k-vector space V. The set of 1-dimensional subspaces of V is denoted $\mathbf{P}(V)$. When $V = k^{N+1}$ we also write $k\mathbf{P}^N$. This notation is usually only employed when $k = \mathbf{R}$ or $k = \mathbf{C}$.

Every non-zero vector in V spans a 1-dimensional subspace. This gives a surjection $V \smallsetminus \{0\} \to \mathbf{P}V$. Two vectors span the same 1-dimensional subspace if and only if one is a non-zero multiple of the other. That is, we get a bijection

$$(V \smallsetminus \{0\})/k^* \simeq \mathbf{P}V$$

where $(V \smallsetminus \{0\})/k^*$ is the set of orbits of the group k^* acting on $V \smallsetminus \{0\}$.

When V has a topology (for example $k = \mathbf{R}$ or $k = \mathbf{C}$) this allows us to put a topology on $\mathbf{P}V$. However, this doesn't explain why this topology is natural.

Suppose $W \subset V$ has codimension 1 and let W' be the translate of W by a vector not in W. We obtain a map $W \simeq W' \to \mathbf{P}V$. Its complement is the natural inclusion $\mathbf{P}W \subset \mathbf{P}V$.

Exercise 49.9. Show that, when $k = \mathbf{R}$ or $k = \mathbf{C}$, the inclusions $W \subset \mathbf{P}V$ constructed above are open embeddings and that they cover $\mathbf{P}V$.

We can weaken the assumption that V be a vector space in this construction, at least when $k = \mathbf{R}$ or $k = \mathbf{C}$. What we really need is for V to be an *cone*. That is V should carry a continuous action of the multiplicative monoid k. The *vertex* of V is $0.V$.

Exercise 49.10. The vertex of V is the same as the fixed locus of k^*.

Exercise 49.11. There is a continuous retraction $V \to 0.V$ sending v to $\lim_{\lambda \to 0} \lambda v$. (Note we are assuming $k = \mathbf{R}$ or $k = \mathbf{C}$ here.)

Define $\mathbf{P}V$ to be the set of lines in V, equivariant closed embeddings $k \to V$.

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[2] The definition makes sense even when A is not noetherian and M is not finitely generated. It is less clear how useful the definition is in this generality, however.

[3] By a zero divisor, we mean an element $a \in A$ such that there is a nonzero $b \in A$ with $ab = 0$.

49.3 Cones in algebraic geometry

49.4 Line bundles

If $\mathbf{P}V$ is supposed to be the space of lines in the vector space V then a map $X \to \mathbf{P}V$ should be a family of lines in V *parameterized by* X. In this section, we make sense of what a "family of lines" is supposed to be.

We give several definitions of a line bundle over a scheme. The first will be familiar to those with background in differential geometry.

Exercise 49.12. Show that the functor $A \mapsto \mathrm{GL}_r(A)$ is representable by an affine scheme. (Hint: Show $\mathrm{Mat}_{m\times n}$ is representable by an affine scheme and then construct GL_r as an principal affine open subscheme of $\mathrm{Mat}_{r\times r}$.)

Definition 49.13. A map $X \times \mathbf{A}^n \to X \times \mathbf{A}^m$ is *linear* if it is of the form $(x,t) \mapsto (x, \lambda(x)t)$ where $\lambda : X \to \mathrm{Mat}_{m\times n}(r)$ is a morphism of schemes.

Exercise 49.14. Give an equivalent definition of linearity for a map $\operatorname{Spec} A \times \mathbf{A}^n \to \operatorname{Spec} A \times \mathbf{A}^m$ in terms of the homomorphism of commutative rings

$$A[t_1, \dots, t_m] \to A[t_1, \dots, t_n].$$

Definition 49.15 (Line bundles via charts)**.** A line bundle on a scheme X is a scheme L and a projection $\pi : L \to X$, together with a cover of X by affine open subschemes $U \subset X$ and isomorphisms $\phi_U : \pi^{-1}U \simeq X \times \mathbf{A}^1$ such that the transition maps

$$(U \cap V) \times \mathbf{A}^1 \xleftarrow{\phi_U|_{U\cap V}} \pi^{-1}(U \cap V) \xrightarrow{\phi_V|_{U\cap V}} (U \cap V) \times \mathbf{A}^1$$

are linear.

Definition 49.16 (Line bundles via the functor of points)**.** A line bundle on a scheme X is a scheme L over X with an action of $\mathbf{A}^1$ on the fibers of L over X that is locally isomorphic in X to the action of $\mathbf{A}^1$ on itself.

Exercise 49.17. Show that these two definitions of line bundles are equivalent.

Reading 49.18. [Vak14, §§5.4, 9.7]

Definition 49.19. Let $A \to B$ be an injective homomorphism of commutative rings. We say that A is *integrally closed* in B if every $x \in B$ that satisfies a monic polynomial with coefficients in A lies in A.

Definition 49.20. A scheme X is said to be *normal* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.

49.5 Torsors under the multiplicative group

Definition 49.21 (Torsor)**.** Let G be an algebraic group. A G-torsor over a scheme X is a G-action on an X-scheme P such that there is a cover of X by open subschemes $U \subset X$ such that $P\big|_U$ is isomorphic to G_U as a G-scheme.

Exercise 49.22. (i) Suppose that L is a line bundle on X. Define $\Phi(L) = \underline{\mathrm{Isom}}(L, \mathbf{A}^1_X)$ to be the sheaf whose value on $U \subset X$ is the set of isomorphisms $L \simeq \mathbf{A}^1_U$. Show that $\underline{\mathrm{Isom}}(L, \mathbf{A}^1_X)$ is a $\mathbf{G}_m$-torsor.

(ii) Suppose that P is a $\mathbf{G}_m$-torsor on X. Define $\Psi(P) = \underline{\mathrm{Hom}}(P, \mathbf{A}^1_X)$ to be the sheaf whose value on $U \subset X$ is the set of $\mathbf{G}_m$-equivariant morphisms $P\big|_U \to \mathbf{A}^1_U$. Show that $\underline{\mathrm{Hom}}(P, \mathbf{P}^1_X)$.

(iii) Show that Φ and Ψ define inverse equivalences of categories between the category of $\mathbf{G}_m$-torsors and the category of line bundles on X.

50 Quasicoherent sheaves on projective space

E Chow's lemma

51 Morphisms to projective space

51.1 Blowing up

51.2 A criterion for closed embeddings

51.3 Ample line bundles

51.4 Another proof of Noether normalization

In this section we will find a more geometric construction of Noether normalization. This yields a slightly less general version of the theorem, but it is just as good for practical purposes.

Exercise 51.1. (i) Let S be a scheme and suppose that M is a $m \times n$ matrix with coefficients in $\Gamma(S, \mathcal{O}_S)$ such that for every point $\xi \in S$ the matrix $M(\xi)$ has rank m. Construct a morphism of S-schemes $\mathbf{P}^m_S \to \mathbf{P}^n_S$ sending $(\mathscr{L}, x)$ to $(\mathscr{L}, xM)$.

(ii) What goes wrong in the previous part when $\operatorname{rank} M < m$?

(iii) More generally, suppose that M is an $m \times n$ matrix as above. Let $U \subset \mathbf{P}^m_S$ be the subfunctor consisting of all $(\mathscr{L}, x) \in \mathbf{P}^m_S$ such that xM generates L. Show that the formula above gives a map $U \to \mathbf{P}^n_S$.

(iv) Show that U is open in $\mathbf{P}^m_S$.

Exercise 51.2. Let k be a field, $p \in \mathbf{P}^n_k$, and $H \subset \mathbf{P}^n_k$ a hyperplane. We make the following construction precise: For any point $q \in \mathbf{P}^n_k \smallsetminus \{p\}$, there is a unique line connecting p and q, denoted $L(p, q)$. This line intersects H in a unique point, hence determines a map $\mathbf{P}^n_k \smallsetminus \{p\} \to \mathbf{P}^{n-1}_k$.

(i) First we explain what we mean by a hyperplane. Fix a linear equation $f(x_0, \ldots, x_n)$. For any k-scheme S, we let $H(S)$ be the set of all $(\mathscr{L}, x_0, \ldots, x_n) \in \mathbf{P}^n_k(S)$ (here $\mathscr{L}$ is an invertible sheaf on S and $x_i \in \Gamma(S, \mathscr{L})$ generate $\mathscr{L}$) such that $f(x_0, \ldots, x_n) = 0$ as an element of $\mathscr{L}$. Show that H is representable by a closed subscheme of $\mathbf{P}^n_k$ and that $H \simeq \mathbf{P}^{n-1}_k$.

(ii) Suppose that p and q are two *disjoint* S-points of $\mathbf{P}^n_k$. Show that there is a unique linear map $g : \mathbf{P}^1_S \to \mathbf{P}^n_k$ such that $g(0) = p$ and $g(\infty) = q$. (Here 0 is the S-point $(\mathcal{O}_S, 0, 1) \in \mathbf{P}^1_S(S)$ and ∞ is the S-point $(\mathcal{O}_S, 1, 0) \in \mathbf{P}^1_S(S)$. 'Linear' means that the map must be of the form $(\mathscr{L}, x_0, x_1) \mapsto (\mathscr{L}, g(x_0, x_1))$ where g is a linear function.)

(iii) Let g and H be as in the last two parts. Show that $g^{-1}H$ consists of a single S-point of $\mathbf{P}^1_S$.

Exercise 51.3. Let $X = \operatorname{Spec} B$ and assume that $f : X \to \mathbf{A}^n_k$ is a finite map that is not surjective. We construct a finite map $X \to \mathbf{A}^{n-1}_k$.

(i) Embed $\mathbf{A}^n_k \subset \mathbf{P}^n_k$ by the map sending $(x_1, \ldots, x_n)$ to $(\mathcal{O}, 1, x_1, \ldots, x_n)$. Show that there is a finite map $\bar{f} : \overline{X} \to \mathbf{P}^n_k$ such that $\bar{f}^{-1}\mathbf{A}^n_k = X$ and $\bar{f}\big|_X = f$. (Hint: Take the 'integral closure' of $\mathbf{P}^n_k$ in X.) (Suggestion: You may want to skip this part of the problem, since it is not necessary to prove Noether normalization if you set up your induction carefully.)

(ii) Choose a point $p \in \mathbf{P}^n_k$ not on H or X.

Bibliography

[AGV 3] M. Artin, A. Grothendieck, and J. Verdier, *SGA4, 1963/64*, Springer Lecture Notes **269, 270, 305** (1972–3).

[AM69] M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra*, vol. 19, Addison-Wesley Reading, MA, 1969.

[ano] anon, *Why is a proper, affine morphism finite?*, MathOverflow, URL:http://mathoverflow.net/q/125793 (version: 2013-03-28).

[Eis91] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, Springer–Verlag, 1991.

[GD67] A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique*, Publications Mathématiques de l'I.H.É.S. (1960–1967).

[GD71] A. Grothendieck and J. A. Dieudonné, *Eléments de gómétrie algébrique I*, Springer–Verlag, 1971.

[Gro57] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Mathematical Journal **9** (1957), no. 3, 119–221.

[Har77] R. Hartshorne, *Algebraic geometry*, vol. 52, Springer, 1977.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory, third edition*, Springer-Verlag.

[Mum99] D. Mumford, *The red book of varieties and schemes: includes the michigan lectures (1974) on curves and their jacobians*, vol. 1358, Springer, 1999.

[Sta15] The Stacks Project Authors, *The stacks project*, `stacks.math.columbia.edu`, 2015.

[Vak14] R. Vakil, *Foundations of algebraic geometry*, 2014, `math.stanford.edu/~vakil/216blog`.