Math 6170: Algebraic geometry

Jonathan Wise

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1 Introduction

This is a course about the modern language of algebraic geometry, the theory of schemes. We will begin where modern algebraic geometry began, with A. Weil in 1949, somewhere near the intersection of complex geometry and number theory.

Consider a polynomial equation, or a collection of polynomial equations, in several variables, with integer coefficients. One may search for solutions to these equations in any ring. The search for solutions in the integers or rationals, for example, is known today as Diophantine geometry; it is a subject as old as mathematics and is still filled with unsolved problems today. Along with Weil, we will be considered with a surprising relationship between two simpler problems: determining the solutions in the finite fields $\mathbf{F}_q$ and in the complex numbers $\mathbf{C}$.

The nature of the solutions in these two settings could not be any more different. Except in a few special cases, the solutions in $\mathbf{C}$ are infinite in number with a non-trivial topology. Over $\mathbf{F}_q$ the set of solutions is a finite set, with no topology in sight.

Yet Weil observed that the topology of the solutions in $\mathbf{C}$ constrained, and is constrained by, the number of solutions over each finite field. Two previously unrelated aspects of the equations were suddenly perceived to be connected. This set off a search for appropriate foundations for algebraic geometry that would enable one to speak in one breath about geometry over the complex numbers and geometry over finite fields.

It took nearly a decade and multiple attempts by some of the greatest mathematicians of the 20th century before an elegant and workable theory coalesced. That was the theory of schemes, announced by A. Grothendieck in his 1958 ICM address and developed by A. Grothendieck and J. Dieudonné in their *Eléments de géométrie algébrique*.[1] Grothendieck later described a scheme as a "magic fan"[2] interpolating between number theory at one extreme and complex geometry at the other. Understanding these schemes will be the subject of this course.

[1] The idea of a scheme is closely related to earlier definitions by Serre, Cartier, Chevalley–Nagata, and Weil.

[2] *éventail magique*

We will begin with Weil's conjectures, which can be appreciated and even rediscovered—though not proved!—with no prior knowledge of algebraic geometry. Thus motivated, we will then lay the foundations for the theory of schemes.

The most important foundational concept is that of a sheaf, to which we will devote a significant amount of time. Not only are sheaves an essential tool in algebraic geometry, but they can offer a valuable perspective in topology and differential geometry as well; we will touch on both of these applications in this class. We will also introduce some homological algebra and category theory.

We won't prove Weil's conjectures in this class, at least not in general: it took more than 15 years and thousands of pages of work for M. Artin, A. Grothendieck, and P. Deligne to do that! But by the end of the course, we will have developed enough machinery to prove Weil's conjectures for algebraic curves. If everything goes to plan, we will conclude the course with that proof.

2 Prerequisites

I intend for this class to be accessible to motivated first-year graduate students. The only truly essential prerequisites are comfort with

> **commutative algebra**—specifically commutative rings, their modules, and notions like localization, homomorphisms, tensor product, etc.—and
>
> **point set topology**.

The more familiar with these subjects you are the more you will be able to get out of this course.

Many concepts in algebraic geometry are motivated by ideas from complex geometry, algebraic topology, differential geometry, and number theory. Familiarity with these subjects is not required for this course, but if you have encountered them before a lot of things in this class will look familiar. It may be particularly helpful if you have encountered **homology** before.

3 Meeting times and office hours

The course meets on MWF at 9am in ECST 1B21. There will be an additional problem session, time TBD, probably on Wednesday afternoons (see Section 6).

Feel free to talk to me in my office at any time. The best time to catch me, at least for the first few weeks of the semester, will be immediately after class.

4 Syllabus

Here is an outline of the topics that I hope[3] to cover in this course, in roughly the order we will encounter them:

[3] of course, this list is provisional

(1) affine space and projective space

(2) examples of Weil's conjectures

(3) introduction to category theory

(4) sheaf theory and sheaf cohomology

(5) affine and projective schemes

(6) fiber products of schemes

(7) divisors, line bundles, and the cohomology of line bundles

(8) algebraic curves and the Riemann–Roch theorem

(9) coherent and quasi-coherent sheaves

(10) the Grassmannian (and other representable functors)

(11) infinitesimal extensions; smooth, unramified, and étale morphisms

(12) differentials, the tangent space, and the cotangent sheaf

(13) proper and separated morphisms

(14) intersection theory on surfaces

(15) proof of Weil's conjectures for curves

5 References

My main reference for this course will be

> Vakil, R. *Foundations of algebraic geometry.* Available online: math.stanford.edu/∼vakil/216blog, 2012.

Many other excellent references exist, including the classic

> Hartshorne, R. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer–Verlag, 1977.

Many people like the following book, although I personally haven't found it as helpful as other references:

> Shafarevich, I. *Basic algebraic geometry*, 1 and 2. Springer–Verlag, 1994.

The following book may also be helpful, although I do not plan to use it while preparing my lecutres:

> Harris, J. *Algebraic geometry, a first course.* Graduate Texts in Mathematics, 133. Springer–Verlag, 1995.

Mumford's "red book" is a very readable classic:

> Mumford, D. *The red book of varieties and schemes.* Lecture Notes in Mathematics, 1358. Springer–Verlag, 1999.

I plan to consult the references below while preparing the course. I don't necessarily encourage you to try to learn from them on the first pass, but you may find it helpful to look at them from time to time.

> Shafarevich, I (ed.). *Algebraic geometry*, I–V. Encyclopedia of Mathematical Sciences, Vol. 35. Springer, 1995.

> Mumford, D. *Lectures on curves on an algebraic surface.* Annals of Mathematics Studies, No. 59. Princeton University Press, 1966.

> Bredon, G. *Sheaf theory.* Graduate Texts in Mathematics, 170. Springer–Verlag, 1977.

> Grothendieck, A. Sur quelques points de algèbre homologique. Tôhoku Math. J. (2) 9 1957 119–221.

> Grothendieck, A. and Dieudonné, J. Eléments de géométrie algébrique. Inst. Hautes Études Sci. Publ. Math. No. 4, 8, 11, 17, 20, 24, 28, 32.

> Godement, R. *Topologie algébrique et théorie de faisceaux.* Actualités Scientifiques et Industrielles, No. 1252. Hermann, 1973.

6 Assignments and evaluation

Of course, engagement through exercises is the only way to learn mathematics. Exercises will be plentiful in this course, and I expect you to do a lot of them. I won't be able to collect and read all of the exercises you do, but you should still do a lot of them!

During the course of the semester we will write notes for the course. I will make some contributions, but I expect you to do the bulk of the writing. Your evaluation in the class will be based *entirely* on your contributions to the notes.

Here are the rules for contributing to the notes:

- Your contributions can be any suitable content that you like. For the most part, however, I expect you to complete numbered exercises in the document, or to fill in the details of unproven statements.
- You should make one contribution by Friday of each week.

- Submit your contributions to me by e-mailing me a tex file. If you haven't learned tex yet, now is the time. I may hand back a paper copy with corrections, which I'll expect you to incorporate into your write-up.

- In order to avoid duplicating work, there will be a limit on the number of people allowed to work on a given problem at a given time. You will have to tell me what you are working on by the Monday before it is due.

- You must cite any resources you consult, including talking to other people!

- Collaborative contributions are allowed as long as you let me know by Monday and I approve.

- **Important!** Each Wendesday there will be a meeting outside of class where you will give a brief presentation of your contribution to the rest of the class. Not everyone will present at each meeting, but you should expect to present at least once every two weeks. I usually will not attend the meeting.

- Towards the end of the semester (perhaps over spring break) you will be asked to make a contribution in which you independently research an outside topic.

7 Illness

According to the CDC, this year's flu season is particularly severe, both in symptoms and extent. If you develop flu-like symptoms, please heed the advice of the campus health center and stay home from class. I would like to add to this a personal request, as the father of an infant for whom even a routine cold could be a severe medical event, that you stay home from class if you are feeling at all ill. I will of course be very happy to help you via e-mail to keep up with course material while you are at home, as well as to help you catch up when you return, should you fall behind.