**Theorem 1** (Chevalley)**.** *Let $\varphi : A \to B$ be a homomorphism of rings making $B$ finitely presented over $A$ and let $Z$ be a constructible subset of* $\operatorname{Spec} B$*. Then $\varphi^*(Z)$ is a constructible subset of* $\operatorname{Spec} A$*.*

1. Show that it is sufficient to assume $B = A[x]$.
2. Show that it is sufficient to assume $Z = V(f_1, \ldots, f_n) \cap D(g)$ for some $f_1, \ldots, f_n, g \in B$.
3. Show that it is sufficient to assume that the $f_i$ all have leading coefficient 1. (Hint: Using induction on the degree of the $f_i$ and on $n$. Let $a$ be the leading coefficient of one of the $f_i$. Then $\operatorname{Spec} A = D(a) \cup V(a)$.)
4. Show that it is sufficient to assume $n = 1$. (Hint: Induct on the degree of the $f_i$. Replace $f_i$ by $f_i - hf_j$ for suitably chosen $i$ and $j \neq i$ and $h \in B$.)
5. Prove that ${\varphi^*}^{-1}(p)$ is nonempty if and only if $g$ is not nilpotent in $\operatorname{Spec} \mathbf{k}(p)[x]/f(x)$.
6. Let $d$ be the degree of $f$. Prove that if $g$ is nilpotent in $\operatorname{Spec} \mathbf{k}(p)[x]/f(x)$ then $g^d = 0$. (Hint: represent multiplication by $g$ as a matrix.)
7. Conclude that ${\varphi^*}^{-1}(p)$ is nonempty if and only if $g^d = 0$ in $\operatorname{Spec} \mathbf{k}(p)[x]/f(x)$.
8. Show that the set of points $p \in \operatorname{Spec} A$ such that $g^d = 0$ in ${\varphi^*}^{-1}(p)$ is closed in $\operatorname{Spec} A$. Conclude that $\varphi^*(V(f) \cap D(g))$ is open.

**Theorem 2.** *Suppose that $K$ is a field and $L$ is a field extension of $K$ that is finitely presented over $K$ as a ring. Then $L$ is an algebraic extension of $K$.*

9. Suppose $x \in L$. Use this to construct a map $\operatorname{Spec} L \to \operatorname{Spec} K[x]$.
10. Show that the image of this map is constructible. Conclude that the image of $\operatorname{Spec} L$ is in $\operatorname{Spec} K[x]$ is a *closed* point.
11. Deduce the theorem.

**Theorem 3.** *Let $K$ be a field and let $V$ be a closed subset of* $\operatorname{Spec} K[x_1, \ldots, x_n]$*. Let $\overline{K}$ be an algebraic closure of $K$. If $V(\overline{K}) = \varnothing$ then $V = \varnothing$.*

12. Prove this theorem from the last one.

**Theorem 4.** *Let $K$ be a field and let $I$ be an ideal of $K[x_1, \ldots, x_n]$. Let $Z$ be the set of points $\xi$ in $\overline{K}^n$ such that $f(\xi) = 0$ for all $f \in I$. Let $J$ be the set of $f \in K[x_1, \ldots, x_n]$ such that $f(Z) = 0$. Then $J = \sqrt{I}$.*

13. Let $g$ be an element of $J$. Show that $V(I) \cap D(J)$ has no $\overline{K}$-points. Conclude that $V(I) \cap D(J) = \varnothing$.
14. Deduce that $g$ is nilpotent in $K[x_1, \ldots, x_n]/I$.
15. Prove that $\sqrt{I} = J$.