1 Square-zero extensions

Definition 1. Let A be a commutative ring and let B be an A-algebra. A *square-zero extension of B* is a pair (B', π) where π an A-algebra surjection $B' \to B$ whose ideal $I \subset B'$ satisfies $I^2 = 0$.

A morphism of square-zero extensions of B from (B', π) to (B'', τ) is a morphism of A-algebras $\varphi : B'' \to B'$ such that $\tau\varphi = \pi$.

Lemma 2. *If $\pi : B' \to B$ is a surjective homomorphism of commutative rings with ideal I, and M is a B-module then M may be regarded as a B'-module by way of π. Conversely, a B'-module M' is induced from a B-module in this way if and only if $IM' = 0$.*

Suppose $\pi : B' \to B$ is a square-zero extension of B. Let $I = \ker(\phi)$. Then $I^2 = 0$, so I can be regarded as a B-module.

Definition 3. Let A be a commutative ring, B an A-algebra, and J a B-module. A *square-zero extension of B by J* is a triple (B', π, σ) where (B', π) is a square-zero extension of B (Definition 1) and $\sigma : J \simeq \ker(\sigma)$ is an isomorphism. A morphism $(B', \pi, \sigma) \to (B'', \tau, \rho)$ of square-zero extensions by J is a morphism of square-zero extensions $\phi : (B', \pi) \to (B'', \tau)$ such that $\phi\sigma = \rho$.

The category of square-zero A-algebra extensions of B by J is denoted $\mathbf{Exal}_A(B, J)$. The set of isomorphism classes of $\mathbf{Exal}_A(B, J)$ is denoted $\mathrm{Exal}_A(B, J)$.

Recall the definition of a derivation:

Definition 4. Suppose that A is a commutative ring, B is an A-algebra, and J is a B-module. An *A-derviation from B into J* is a function $\delta : B \to J$ with the following properties:

(i) For all b and c in B, we have $\delta(b + c) = \delta(b) + \delta(c)$.

(ii) For all $a \in A$, we have $\delta(a) = 0$.

(iii) For all b and c in B, we have $\delta(bc) = b\delta(c) + c\delta(b)$.

The set of A-derivations from B into J is denoted $\mathrm{Der}_A(B, J)$.

2 Module structure

Suppose that A is a commutative ring, B is an A-algebra, and J is a B-module. Let $B + \epsilon J$ be the set of all symbols $b + \epsilon x$ where $b \in B$ and $\epsilon \in J$. Give $B + \epsilon J$ a ring structure:

$$\begin{aligned} 0 &= 0 + 0\epsilon \\ 1 &= 1 + 0\epsilon \\ (b + \epsilon x) + (c + \epsilon y) &= (b + c) + \epsilon(x + y) \\ (b + \epsilon x)(c + \epsilon y) &= bc + \epsilon(by + cx) \end{aligned}$$

Definition 5. Set $\pi(b + \epsilon x) = b$ and $\sigma(x) = 0 + \epsilon x$ for all $b \in B$ and $x \in J$. Then $(B + \epsilon J, \pi, \sigma)$ is a square-zero extension of B by J. This is called the *trivial square-zero extension of B by J*.

Lemma 6. *Suppose that (B_1, π_1, σ_1) and (B_2, π_2, σ_2) are two square-zero A-algebra extensions of B by J_1 and J_2, respectively. Let*

$$B_1 \times_B B_2 = \{(b_1, b_2) \in B_1 \times B_2 \mid \pi_1(b_1) = \pi_2(b_2)\}.$$

Then $B_1 \times_B B_2$ is a subring of $B_1 \times B_2$. Set $\pi(b_1, b_2) = \pi_1(b_1) = \pi_2(b_2)$ and set $\sigma(x_1, x_2) = (\sigma_1(x_1), \sigma_2(x_2))$ for $x_i \in J_i$. Then $(B_1 \times_B B_2, \pi, \sigma)$ is a square-zero extension of B by $J_1 \times J_2$.

This determines a functor:

$$\mathbf{Exal}_A(B, J_1) \times \mathbf{Exal}_A(B, J_2) \to \mathbf{Exal}_A(B, J_1 \times J_2)$$

Lemma 7. *Suppose that (B', π, σ) is a square-zero extension of B by J and that $\phi : J \to K$ is a homomorphism of B-modules. Let*

$$B'' = (B' + \epsilon K)/(\sigma, -\phi)J$$

Define $\tau : B'' \to B$ by $\tau(b' + \epsilon x) = \pi(b')$ and define $\rho : K \to B''$ by $\rho(x) = 0 + \epsilon x$. Then (B'', τ, ρ) is an object of $\mathbf{Exal}_A(B, K)$ and this determines a functor:

$$\phi_* : \mathbf{Exal}_A(B, J) \to \mathbf{Exal}_A(B, K)$$

Lemma 8. *Suppose that A is a commutative ring, B is an A-algebra, and that J and K are B-modules. Let $p : J \times K \to J$ and $q : J \times K \to K$ be the two projections. Then the functor*

$$(p_*, q_*) : \mathbf{Exal}_A(B, J \times K) \to \mathbf{Exal}_A(B, J) \times \mathbf{Exal}_A(B, J)$$

is an equivalence of categories.

If B' and B'' are square-zero A-algebra extensions of B by J define $B' + B''$ to be the image of the pair (B', B'') under the sequence of maps

$$\mathbf{Exal}_A(B, J) \times \mathbf{Exal}_A(B, J) \xleftarrow{(p_*, q_*)} \mathbf{Exal}_A(B, J \times J) \xrightarrow{\Sigma_*} \mathbf{Exal}_A(B, J)$$

where $\Sigma : J \times J \to J$ is the map $\Sigma(x, y) = x + y$. If $b \in B$ let $\mu : J \to J$ be the map $\mu(x) = b.x$. For any $B' \in \mathbf{Exal}_A(B, J)$, define $b.B'$ to be the image of B' under

$$\mathbf{Exal}_A(B, J) \xrightarrow{\mu_*} \mathbf{Exal}_A(B, J).$$

Lemma 9. *These constructions make* $\mathrm{Exal}_A(B, J)$ *into a B-module.*

3 Exact sequence

Our goal is the following theorem:

Theorem 10. *Suppose that there is a sequence of homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ and J is a C-module. We can also regard J as a B-module via the morphism $B \to C$. There is an exact sequence:*

$$0 \to \mathrm{Der}_B(C, J) \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J) \to \mathrm{Exal}_A(C, J) \to \mathrm{Exal}_A(B, J)$$

Here are the definitions of the maps in the sequence:

(i)

(ii)

(iii) $\mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J)$. Suppose that $\delta : B \to J$ is a derivation. Then $\mathrm{id} + \epsilon\delta : B \to C + \epsilon J$ is a ring homomorphism. Then $\mathrm{id} + \epsilon\delta$ makes $C + \epsilon J$ into a square-zero B-algebra extension of C by J.

(iv) $\mathrm{Exal}_B(C, J) \to \mathrm{Exal}_A(C, J)$.

(v) $\mathrm{Exal}_A(C, J) \to \mathrm{Exal}_A(B, J)$. Suppose that (C', π) is a square-zero extension of C. Let

$$B' = C' \times_C B = \{(c, b) \mid g(b) = \pi(c)\}.$$

Then B' is a square-zero extension of B. If C' is a square-zero extension of C by J then B' is a square-zero extension of B by J.

4 Cotangent complex

Lemma 11. *Suppose that $A \to B$ is a surjective homomorphism of commutative rings with ideal I and that J is a B-module. Then*

$$\mathrm{Der}_A(B, J) = 0\,\mathrm{Exal}_A(B, J) \qquad\qquad \simeq \mathrm{Hom}_{B\text{-}\mathbf{Mod}}(I/I^2, J).$$

Lemma 12. *Suppose that B is a free A-algebra, generated by a set S. Then*

$$\mathrm{Der}_A(B, J) \simeq \mathrm{Hom}_{B\text{-}\mathbf{Mod}}(BS, J) = J^S$$
$$\mathrm{Exal}_A(B, J) = 0$$

where BS denotes the free B-module generated by S.

Theorem 13. *Let A be a commutative ring, let C be an A-algebra, and let J be a C-module. Suppose that B is a free A-algebra and that $B \to C$ is a surjection with kernel I. Let $L^\bullet$ be the complex*

$$L^\bullet = [I/I^2 \xrightarrow{d} \Omega_{B/A}]$$

in degrees $[-1, 0]$. Then

$$\mathrm{Der}_A(C, J) \simeq H^0(\mathrm{Hom}(L^\bullet, J))$$
$$\mathrm{Exal}_A(C, J) \simeq H^1(\mathrm{Hom}(L^\bullet, J))$$