# Math 52 – Spring 2012

## Assignment #9

**Problem 1.** In this problem we will compute the gravitational force field associated to some regions of non-constant density.

(a) At up to a constant multiple, the gravitational field associated to a point of mass $m$ at position $(x, y, z)$ is

$$F(x_0, y_0, z_0) = m\Big(\frac{x - x_0}{r^3}, \frac{y - y_0}{r^3}, \frac{z - z_0}{r^3}\Big).$$

where $r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$. Show that for any closed surface $S$ containing $(x, y, z)$ we have

$$\int_S F(x_0, y_0, z_0) \cdot (dy_0\, dz_0, dz_0\, dx_0, dx_0\, dy_0) = 4\pi m.$$

Electrical field is also an inverse square field, so it behaves in exactly the same way.

(b) Suppose that a closed surface $S$ contains point $P_1, \ldots, P_n$ of masses $m_1, \ldots, m_n$. Relate

$$\int_S F(x_0, y_0, z_0) \cdot (dy_0\, dz_0, dz_0\, dx_0, dx_0\, dy_0)$$

to the total mass $\sum_{i=1}^n m_i$ inside the region.

(c) Demonstrate that for any nice vector field $F$,

$$\operatorname{div}(F)(x_0, y_0, z_0) = \lim_{r \to 0} \int_{S_r} F \cdot \mathbf{n}\, dA$$

where $S_r$ is a sphere of radius $r$ around the point $(x_0, y_0, z_0)$ and $\mathbf{n}$ is a unit normal vector to the surface of the sphere (pointing away from $(x, y, z)$), and $dA$ is infinitesimal surface area on the sphere.

(d) Suppose $R$ is a 3-dimensional region of non-uniform density $\rho$ (that is, $\rho(x, y, z)$ is the density of $R$ at the point $(x, y, z)$). Using (a), give a volume integral depending on $(x_0, y_0, z_0)$ whose value is the gravitational force experienced by a point with coordinates $(x_0, y_0, z_0)$. This is a vector field on 3-dimensional space whose value at $(x_0, y_0, z_0)$ will be denoted by $F(x_0, y_0, z_0)$.

(e) Compute $\operatorname{div}(F)$. (Use part (c).)

(f) Suppose that $R$ is a region of total mass $m$ contained in a sphere $S$. If $(x_0, y_0, z_0)$ is a point outside of $S$, write down an iterated integral for $F(x_0, y_0, z_0)$. *Do not under any circumstances evaluate this integral.*

(g) Use symmetry to determine which direction the field $F(x_0, y_0, z_0)$ from part (f) is pointing when $R$ is *spherically symmetric*. (Hint: your answer should involve the center of mass of $R$.)

(h) Use the divergence theorem and (g) to evaluate $F(x_0, y_0, z_0)$ *without doing the iterated integral*.

(i) Suppose that $R$ is a "3-dimensional annulus", given by the equation $1 \leq x^2 + y^2 + z^2 \leq 4$, with constant density $\rho = 1$. What is $F(x_0, y_0, z_0)$ if $(x_0, y_0, z_0)$ is contained inside the sphere of radius 1? (This should be pretty direct from your calculation in part (f).)

**Problem 2.** In the last problem we saw that divergence is related to the limit of a surface integral. Formulate a similar statement relating curl to a path integral.

**Problem 3.** Using Green's theorem, we have seen that it is possible to compute the area of a 2-dimensional region in $\mathbf{R}^2$ as a line integral $\int_C x\, dy$ around the boundary.

(a) Is it possible to compute the volume of a region $R$ as a surface integral over its boundary? If so, say what the integrand is; if not, say why not.

(b) Is it possible to compute the surface area of a surface in 3-dimensional space as an integral around its boundary? If so, say what the integrand is; if not, say why not.

**Problem 4.** (a) Suppose a string of length 1 is cut at one place, chosen uniformly at random along the string. What is the expected length of the shorter piece of string that results?

(b) Suppose a string of length 1 is cut at two places, chosen uniformly at random. What is the expected length of the piece of string of intermediate length? (Actually computing the integrals involved here isn't hard, but it might take some time. You'll get all of the benefit of this problem just by setting up the integrals.)

**Problem 5.** Differential forms in 3-variables are sums and products symbols of the form $df$, where $f$ is a (nice) function. Here are the rules for manipulating differential forms:

$$df\, dg = -dg\, df$$
$$(df)^2 = 0 \qquad \text{(this actually comes from the line above)}$$
$$d(fg) = f\, dg + g\, df$$
$$df = \text{grad}(f) \cdot (dx, dy, dz).$$

(a) Using the rules above, show that $\text{curl}(F) \cdot (dy\, dz, dz\, dx, dx\, dy) = d(F \cdot (dx, dy, dz))$ for any (nice) vector field $F$.

(b) Show that $\operatorname{div}(F \cdot (dx, dy, dz)) = \operatorname{div}(F)\, dx\, dy\, dz$.

**Problem 6.** Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = \left( \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Define vector fields

$$\begin{aligned} \mathbf{G}_1 &= \mathbf{F}(x, y, z - 1) \\ \mathbf{G}_2 &= \mathbf{F}(x, y, z - 3) \\ \mathbf{G} &= \mathbf{G}_1 + \mathbf{G}_2. \end{aligned}$$

Notice that $\mathbf{G}_1$ is $\mathbf{F}$ shifted by $(0, 0, 1)$ and $\mathbf{G}_2$ is $\mathbf{F}$ shifted by $(0, 0, 3)$.

(a) Let $Q$ be the disc described by the inequality $x^2 + y^2 \leq 1$ and the equation $z = 0$, with its orientation $\mathbf{n}_Q$ pointing in the direction of the positive $z$-axis. Compute $\int_Q \mathbf{G} \cdot \mathbf{n}_Q \, dA$ (where $dA$ is infinitesimal surface area).

(b) Let $R$ be the piece of the paraboloid $z = 2 - 2(x^2 + y^2) \geq 0$. Compute $\int_R \mathbf{G} \cdot \mathbf{n}_R \, dA$, where $\mathbf{n}_R$ is the unit normal vector of $R$ with positive $z$-component and $dA$ is infinitesimal surface area. (Hint: compute $\int_{Q+R} \mathbf{G} \cdot \mathbf{n} \, dA$ and then use your answer from the last part.)

(c) Let $S$ be the piece of the paraboloid $z = 4 - 4(x^2 + y^2) \geq 0$ with orientation $\mathbf{n}_S$ having positive $z$-component. Compute $\int_S \mathbf{G} \cdot \mathbf{n} \, dA$.

**Problem 7.** Suppose a toothpick is dropped at random so that its center lies inside a square region $R$. Assume that the sides of $R$ and the toothpick both have length 1. Let $(x, y)$ represent the coordinates where the center of the toothpick falls, and let $\theta$ represent the angle that the toothpick makes with a horizontal line through its center.

(a) If $\theta$ is a fixed value, describe all possible positions where the center of the toothpick could land so that the entire toothpick is inside the rectangle. It may help to do a few examples, like $\theta = 0$, $\theta = \frac{\pi}{3}$, $\theta = \frac{\pi}{4}$, etc. to get a feeling for this question.

(b) Let $P$ be the 3-dimensional region representaing all possible outcomes $(x, y, \theta)$ with the center of the toothpick in $P$. Let $Q$ denote the 3-dimensional region (in coordinates, $(x, y, \theta)$) representing the positions and angles where the toothpick crosses the boundary of $R$. Find $E(Q|P)$.

(c) What is the probability that the toothpick does not cross the boundary?

**Problem 8.** Suppose that $\mathbf{F}$ is a vector field in 3-dimensional space that is nice (meaning its components have all partial derivatives of all orders) at all points except on the lines $x - y = z - 1 = 0$ and $x + y = z + 1 = 0$. Assume that

(i) $\operatorname{curl}(\mathbf{F}) = 0$,

(ii) $\int_C \mathbf{F} \cdot d\mathbf{r} = 4$ when $C$ is the curve with parameterization,

$$\begin{aligned} x(t) &= 0 \\ y(t) &= \cos(t) \qquad\qquad 0 \le t \le 2\pi \\ z(t) &= \sin(t) + 1, \end{aligned}$$

(iii) $\int_D \mathbf{F} \cdot d\mathbf{r} = -1$ when $D$ is the curve with parameterization

$$\begin{aligned} x(t) &= 3\cos(t) \\ y(t) &= 3\sin(t) \qquad\qquad 0 \le t \le 2\pi \\ z(t) &= 0. \end{aligned}$$

Compute $\int_E \mathbf{F} \cdot d\mathbf{r}$ where $E$ is the curve

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= 0 \qquad\qquad 0 \le t \le 2\pi \\ z(t) &= \sin(t) - 1. \end{aligned}$$