Math 52 – Spring 2012

Assignment #8

To harmonize the notation of the textbook with the notation from class:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \int_S \mathbf{F}\cdot(dy\,dz, dz\,dx, dx\,dy)$$
$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot(dx, dy, dz).$$

In the above, dS is surface area (what we have usually written dA_S in class), ds is arc length (what we have sometimes called dL_C), $\mathbf{n}$ is a unit normal vector to the surface S, and $\mathbf{T}$ is a unit tangent vector to the curve C. An infinitesimal version of the above translation is

$$\mathbf{n}\,dS = (dy\,dz, dz\,dx, dx\,dy)$$
$$\mathbf{T}\,ds = (dx, dy, dz).$$

Problem 1. §14.5, #14

Problem 2. §14.5, #16

Problem 3. §14.5, #24

Problem 4. §14.6, #2

Problem 5. §14.6, #8

Problem 6. §14.6, #21

Problem 7. §14.7, #8

Problem 8. §14.7, #17 (part (b) of this problem may be a little misleading; we will discuss it in class)

Problem 9. For which of the following vector fields F is $F = \mathrm{grad}(f)$ for some function f? Give a brief justification of your answer.

(a) $F = (yz, xz, xy)$

(b) $F = (x, -z, y)$

(c) $F = \left(\dfrac{z}{x^2+y^2}\ ,\ 0\ ,\ \dfrac{-x}{x^2+y^2}\right)$

(d) $F = \left(\dfrac{z-y}{(x-z)^2+(y-z)^2}, \dfrac{x-z}{(x-z)^2+(y-z)^2}, \dfrac{y-x}{(x-z)^2+(y-z)^2}\right)$

Problem 10. For which of the following vector fields F is $F = \mathrm{curl}(G)$ for some vector field G?

(a) $F = (x, y, z)$

(b) $F = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$ (here $r = \sqrt{x^2 + y^2 + z^2}$)

Problem 11. Let L be the line parameterized by $(x, y, z) = (1, -1, 2) + t(0, 1, 0)$ for $t \in \mathbf{R}$. Find a vector field F such that $\int_C F \cdot (dx, dy, dz)$ is the number of times C circles L counterclockwise when you look in the direction $(0, -1, 0)$.

Problem 12. (extra credit) Let F be the vector field

$$\begin{aligned} F &= \left(\frac{xz}{r(r-2)^2 + rz^2}, \frac{yz}{r(r-2)^2 + rz^2}, -\frac{r-2}{(r-2)^2 + z^2}\right) \\ &= \frac{1}{(r-2)^2 + z^2}\left(\frac{xz}{r}, \frac{yz}{r}, -(r-2)\right) \end{aligned}$$

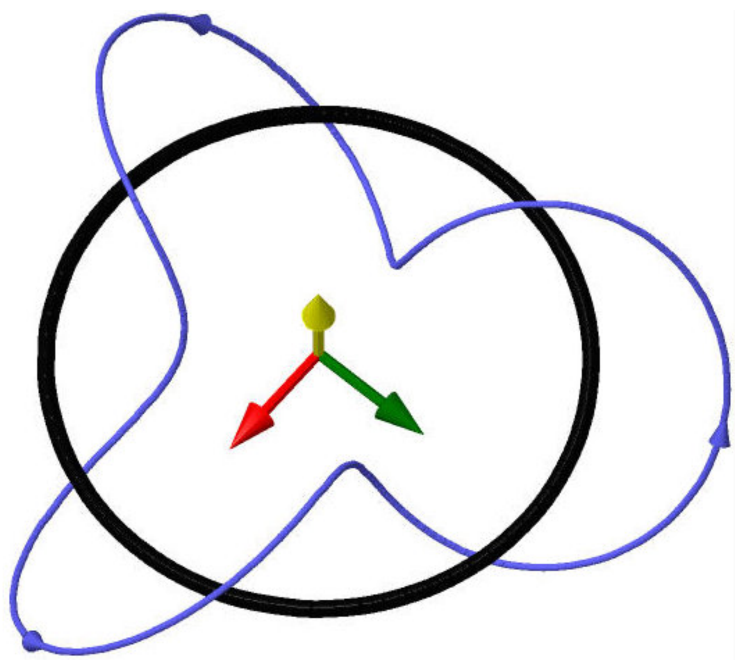
where $r = \sqrt{x^2 + y^2}$. Define $F(0, 0, z) = (0, 0, \frac{2}{4+z^2})$ for all z.

(a) Describe the points where F fails to be defined. (For extra credit, you may describe, with justification, where F fails to be continuous.)

(b) Compute $\mathrm{curl}(F)$.

(c) Compute $\int_C F \cdot (dx, dy, dz)$ when C is the circle defined by the equations $(x-2)^2 + z^2 = 1$ and $y = 0$, given the orientation that is counterclockwise when viewed from the positive y-axis. (Hint: you can save a lot of work by noticing that $r = x$ on the curve C.)

(d) If D is the blue curve depicted below, what is $\int_D F \cdot (dx, dy, dz)$?



In the picture, the black circle has equations $x^2 + y^2 = 2$ and $z = 0$. The x-axis is shown in red, the y-axis in green, and the z-axis in yellow. The orientation of D is indicated by the arrows. You can get more visualizations here: http://nt.sagenb.org/home/pub/169/.

(e) Is F the gradient of a vector field? How do you know?