Math 52 – Spring 2012

Assignment #6

Problem 1. *§14.2, #4*

Problem 2. *§14.2, #10*

Problem 3. *§14.2, #36*

Problem 4. *In appropriate units, the gravitational force of a point mass at the origin on an object at position* (x, y) *of fixed mass is* $F = \frac{1}{r^3}(-x, -y)$*, where* $r = \sqrt{x^2 + y^2}$.

(a) *Find a function* f *such that* $F = \mathrm{grad}(f)$.

(b) *Deduce that* F *is conservative.*

(c) *Compute the work involved in moving a particle from distance* a *away from the origin to a distance* b.

(d) *Explain (in a conceptual way) why your answer in the last part depended only on the radial motion and not on the angular motion.*

Problem 5. *Suppose that a vector field* F *is defined on the whole plane except at the origin. Assume that* $\mathrm{curl}(F) = 0$ *and* $\int_C F \cdot (dx, dy) = 0$ *where* C *is a circle of radius* 1 *around the origin.*

(a) *Is* F *conservative? Justify your answer by explaining why* $\int_C F \cdot (dx, dy) = 0$ *for all closed curves* C. *Be careful to fully explain why the integral is zero for* all *curves encircling the origin.*

(b) *Describe a function* f *that is "nice" at all points of the plane except possibly at the origin such that* $\mathrm{grad}(f) = F$. *Your description can involve* 1*-variable integrals (but should not involve any path integrals or other multivariable concepts).*

(c) *Is your function* f *well-defined at the origin? (Hint: it might help to look at the last problem.)*

Problem 6. *Recall that the sum of vector fields* $F = (a_1, b_1)$ *and* $G = (a_2, b_2)$ *is* $F + G = (a_1 + a_2, b_1 + b_2)$. *The sum of two oriented curves* C *and* D *is the curve* $C + D$ *obtained by traversing one and then the other. The negative* $-C$ *of an oriented curve* C *is the curve obtained by traversing* C *in reverse.*

Suppose that F *and* G *are vector fields and* C *and* D *are oriented curves. Demonstrate the following properties of line integrals:*

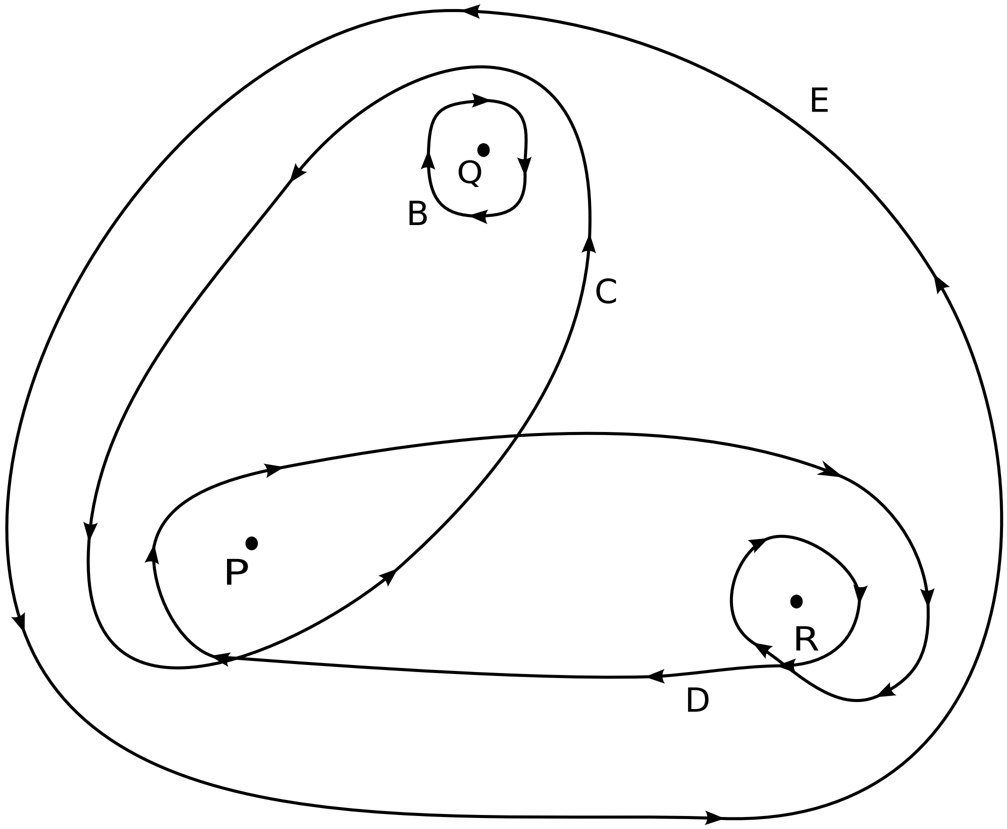
(a) $\int_C (F + G) \cdot (dx, dy) = \int_C F \cdot (dx, dy) + \int_C G \cdot (dx, dy)$

(b) $\int_{C+D} F \cdot (dx, dy) = \int_C F \cdot (dx, dy) + \int_D F \cdot (dx, dy)$ *(you may assume that the endpoint of* C *is the starting point of* D *if you want)*

(c) $\int_{-C} F \cdot (dx, dy) = -\int_C F \cdot (dx, dy)$

You will probably want to reduce these facts to similar facts concerning 1*-variable integrals, which you may take for granted.*

Problem 7. *In the image below, P, Q, and R are points in the plane and B, C, D, and E are closed curves.*



Assume that F is a vector field that is defined at all points of the plane except possibly at P, Q, and R and that $\operatorname{curl}(F) = 0$. *Suppose that*

(i) $\int_B F \cdot (dx, dy) = 1$,

(ii) $\int_C F \cdot (dx, dy) = 2$,

(iii) $\int_D F \cdot (dx, dy) = 3$.

Compute $\int_E F \cdot (dx, dy)$.

Problem 8. *(a) Find a vector field F on the plane that has* all *of the following properties:*

- *(i)* $\operatorname{curl}(F) = 0$,
- *(ii) F is defined and infinitely differentiable away from the points* $(0,0)$ *and* $(2,0)$,
- *(iii)* $\int_C F \cdot (dx, dy) \neq 0$ *if C is the circle* $x^2 + y^2 = 1$,
- *(iv)* $\int_C F \cdot (dx, dy) \neq 0$ *if C is the circle* $(x-2)^2 + y^2 = 1$, *but*
- *(v)* $\int_C F \cdot (dx, dy) = 0$ *if C is the cricle* $(x-1)^2 + y^2 = 4$.

(b) Is F conservative on the region R consisting of all points except $(0,0)$ *and* $(2,0)$*? Justify your answer.*

(c) Is F conservative on the annulus $3 \leq (x-1)^2 + y^2 \leq 5$*? Justify your answer.*
(Hint: you may want to make use of Problem 6 (a).)

Problem 9. *Compute*

$$\int_C \Big((x^2 + y^2)\, dx - 2xy\, dy \Big)$$

where C is the boundary of the triangle with $x \geq 0$, $y \geq 0$, and $x + y \leq 1$ (oriented counterclockwise). (This problem is the same as §14.4, #2.)

Problem 10. *(extra credit) Describe* all *vector fields F such that*

(i) F *is "nice" at all points of the plane except the origin,*

(ii) $\mathrm{curl}(F) = 0$,

(iii) $\int_C F \cdot (dx, dy) = 2\pi$ *when C is a circle around the origin oriented counterclockwise.*

(Hint: consider the difference between two such vector fields.)