Math 52 – Spring 2012

Assignment #3

You may use a symbolic integrator on all of the problems below *except* Problem 12.

Problem 1. *(a) Describe all representations in spherical coordiantes of the point with cylindrical coordinates* $(r, \theta, z) = (1, 0, 1)$.

(b) Describe all representations in cylindrical coordinates of the points with spherical coordinates $(\rho, \phi, \theta) = (2, \frac{\pi}{6}, \frac{\pi}{4})$.

(c) Describe all representations in spherical coordinates of the point with cartesian coordinates $(x, y, z) = (0, 0, 1)$.

(d) Describe all representations in cylindrical coordinates of the point with spherical coordinates $(\rho, \phi, \theta) = (-2, 0, \sqrt{e})$.

Problem 2. *Consider the the right circular cone obtained by taking the triangle with vertices* $(0, 0, 0)$, $(0, 0, 1)$, *and* $(1, 0, 1)$ *by an angle of* 2π *around the* z*-axis. Give inequalities describing the cone in (i) cartesian coordinates, (ii) cylindrical coordinates, and (iii) spherical coordinates.*

Problem 3. §*13.4, #6*

Problem 4. §*13.4, #24*

Problem 5. *Compute the centroid of a right circular cone of radius* a *and height* h. *(Recall that this shape is obtained by rotating a right triangle of base* a *and height* h *by an angle of* 2π *around the axis containing the side of length* h.*)*

Problem 6. §*13.7, #25 (The solution can be found in the back of the book, so the important thing here is to set things up properly.)*

Problem 7. *Let* R *be the disc defined in polar coordinates by* $0 \le r \le 1$ *and* $0 \le \theta \le 2\pi$. *Compute the expected values*

(a) $E(r|R)$ *and*

(b) $E(\theta|R)$.

Is it true that the polar coordinates of the centroid of R *are* $(E(r|R), E(\theta|R))$*?*

Problem 8. *Fix a positive number* b. *Let* R *be the region defined in cartesian coordinates by* $x^2 + y^2 + z^2 \le b^2$ *and* $(x - b)^2 + y^2 + z^2 \ge b^2$. *Find the volume of* R *using integration in spherical coordinates. (Hint: it might help to relabel the cartesian coordinates before switching to spherical coordinates.)*

Problem 9. *The problem that was listed here has been postponed to the next assignment.*

Problem 10. *§13.9, #14*

Problem 11. *Pappus's theorem. Suppose that Q is a 3-dimensional region that is obtained by rotation a 2-dimensional region R inside the xy-plane around the y-axis. For each value of x, let $h(x)$ be the height of the slice of R with that x-coordinate.*

(a) *Explain why* $\int_{x=0}^{\infty} h(x)\,dx = \mathrm{area}(R)$.

(b) *Explain why* $2\pi \int_{r=0}^{\infty} h(r)\,r\,dr = \mathrm{volume}(Q)$ *(think about cylindrical coordinates).*

(c) *Conclude that* $\mathrm{volume}(Q) = 2\pi\, E(x|R)\,\mathrm{area}(R)$.

Problem 12. *If R is a 1-dimensional region with a density function δ then the n-the moment of R is $\int_R x^n\,\delta(x)\,dL$. Using the E notation introduced in class, the n-th moment is* $E(x^n|R)\,\mathrm{mass}(R)$. *The first moment is therefore closely related to the expected value of x (or the center of mass); the second moment is related to the* variance *and* standard deviation *and is sometimes called the* moment of inertia *in physical contexts. You can read more about the significance of the higher moments on Wikipedia.*

Let $\delta(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$ *and let R be the region $x \geq 0$.*

(a) *Explain why the n-th moment of R is the same as $E(x^n|R)$.*

(b) *Compute $E(x|R)$. (Hint: use substitution.)*

(c) *Using integration by parts, show that $E(x^n|R) = \frac{n-1}{2} E(x^{n-2}|R)$ for every integer $n \neq 1$.*

(d) *Show that*

$$E(x^n|R)^2 = \left(\frac{1}{2^n\sqrt{\pi}} \int_0^{\pi} \sin(\theta)^n\,d\theta\right) E(x^{2n+1}|R).$$

(Hint: Use polar coordinates. You may want to use the fact that $2\sin(\theta)\cos(\theta) = \sin(2\theta)$.*)*

(e) *Use your calculations from parts (b), (c), and (d) to compute* $\int_0^{\pi} \sin(\theta)^n\,d\theta$ *for $n = 2, 3, 4$.*

You may use the calculations we've done in class. You may want to check your calculations using a symbolic integrator.