Math 52 – Spring 2012
Exam #3

No materials except pen or pencil and paper. Make sure to explain your answers fully.

First name:

Last name:

Student number:

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Problem 3		/10
Problem 4		/10
Problem 5		/10
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Total		/ 100

Problem 1. (8 points) Let C be the path

$$\begin{aligned} x(t) &= \cos(2\pi t) \\ y(t) &= \sin(2\pi t) \\ z(t) &= t \end{aligned}$$

Compute the length of the path traversed as t increases from 0 to 1.

Solution. We have

$$\begin{aligned} \frac{dx}{dt} &= -2\pi\sin(2\pi t) = -2\pi y \\ \frac{dy}{dt} &= 2\pi\cos(2\pi t) = 2\pi x \\ \frac{dz}{dt} &= 1 \end{aligned}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 8\pi^2 + 1$$

The arc length is

$$\begin{aligned} \int_{t=0}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt &= \int_{t=0}^{1} \sqrt{4\pi^2\sin(2\pi t)^2 + 4\pi^2\cos(2\pi t)^2 + 1}\, dt \\ &= \sqrt{4\pi^2+1}. \end{aligned}$$

$\square$

Problem 2. (10 points) Let R be the square defined by the inequalities $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Let $f(x,y) = x + y - 1$. Suppose that δ is a function on R such that $2 \leq \delta(x,y) \leq 4$ for all values of x and y. What is the largest possible value of $\int_R \delta f \, dA$?

Solution. Notice that $f(x,y) \geq 0$ for $x+y \geq 1$ and $f(x,y) \leq 0$ for $x+y \leq 1$. Therefore we want to maximize δ when $x+y \geq 1$ and minimize δ when $x+y \leq 1$. Let S be the triangle where $x+y \leq 1$ inside R and let T be the triangle where $x+y \geq 1$ inside R. We have

$$
\begin{aligned}
\int_R \delta f \, dA &= \int_S \delta f \, dA + \int_T \delta f \, dA \\
&\leq \int_S 2f \, dA + \int_T 4f \, dA \\
&= \int_R 4f \, dA - \int_S 2f \, dA && \text{just to make calculation simpler,} \\
& && \text{using } S + T = R \\
&= 0 - 2\int_{x=0}^{1}\int_{y=0}^{1-x} (x+y-1)\, dy\, dx && \int_R f \, dA = 0 \text{ by symmetry} \\
& && \text{across the line } x+y=1 \\
&= -2\int_{x=0}^{1} \left((x-1)(1-x) + \frac{(1-x)^2}{2}\right) dx \\
&= -2\int_{x=0}^{1} \left(-\frac{(1-x)^2}{2}\right) dx \\
&= \int_{x=0}^{1} (1-x)^2 dx \\
&= -\frac{(1-x)^3}{3}\bigg|_{x=0}^{1} \\
&= \frac{1}{3}.
\end{aligned}
$$

□

Problem 3. (10 points) Consider the vector field

$$\mathbf{F} = \left(\sqrt{x^3+x} + y^2, e^{-y^2} + 3x\right).$$

Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the boundary of the rectangle $[0,2] \times [0,1]$ oriented counterclockwise. (Hint: $\sqrt{x^3+x}\,dx$ and $e^{-y^2}\,dy$ do not have closed form antiderivatives.)

Solution.

$$\operatorname{curl}(\mathbf{F}) = 3 - 2y.$$

By Green's theorem, we can get the contour integral by integrating the curl over the interior: let R be the rectangle $[0,2] \times [0,1]$, so $C = \partial R$. We have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_R \operatorname{curl}(\mathbf{F})\,dA \\
&= \int_R (3-2y)\,dA \\
&= \int_{x=0}^{2} \int_{y=0}^{1} (3-2y)dy\,dx \\
&= 2(3y - y^2)\Big|_{y=0}^{1} \\
&= 2(3-1) = 4.
\end{aligned}$$

$\square$

Problem 4. (10 points) Rewrite the integral

$$\int_{x=-2}^{3} \int_{y=x^2-3}^{x+3} f(x,y)\, dy\, dx$$

as an integral (or sum of integrals) of the form

$$\int_{y=?}^{?} \int_{x=?}^{?} f(x,y)\, dx\, dy.$$

Solution. The region is

$$\begin{aligned} -2 \le x &\le 3 \\ x^2 - 3 \le y &\le x + 3. \end{aligned}$$

Notice that the inequality $x^2 - 3 \le y$ can be rewritten as $-\sqrt{y+3} \le x \le \sqrt{y+3}$. The inequality $y \le x+3$ can also be rewritten as $x \ge y - 3$.

Here is a picture of the region:

The integral will be made up of two parts because there are two kinds of hoizontal slices. The change between the two occurs at the lower intersection point of the lines $y = x^2 - 3$ and $y = x + 3$. We calculate these intersection points:

$$\begin{aligned} x^2 - 3 &= y = x + 3 \\ x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 0. \end{aligned}$$

Therefore the x-coordinates of the intersection points are -2 and 3. Substituting into $y = x + 3$ we get the y-coordinates. The intersection points are therefore $(-2, 1)$ and $(3, 6)$.

The lowest possible value of y is the same as the lowest possible value of $x^2 - 3$: it is -3. We can now write our region in two pieces:

$$\begin{aligned} -3 \le y &\le 1 \\ -\sqrt{y+3} \le x &\le \sqrt{y+3} \end{aligned} \qquad\qquad \begin{aligned} 1 \le y &\le 6 \\ y - 3 \le x &\le \sqrt{y+3}. \end{aligned}$$

The integral is therefore made of two parts:

$$\int_{y=-3}^{1} \int_{x=-\sqrt{y+3}}^{\sqrt{y+3}} f(x,y)\, dx\, dy + \int_{y=1}^{6} \int_{x=y-3}^{\sqrt{y+3}} f(x,y)\, dx\, dy.$$

$\square$

Problem 5. (10 points) Let R be the 3-dimensional solid region defined by the inequalities

$$x^2 + \frac{y^2}{4} \leq z \leq 6x + y.$$

Compute the volume of R. (Hint: first make the change of coordinates $u = x - 3$, $v = \frac{y}{2} - 1$, and $w = z - 6x - y + 10$, then use cylindrical coordinates.)

Solution. Rearranging the inequality, it becomes

$$x^2 - 6x + \frac{y^2}{4} - y \leq z - 6x - y \leq 0.$$

Complete the square:

$$(x-3)^2 + (\frac{y}{2} - 1)^2 \leq z - 6x - y + 10 \leq 10.$$

In the new coordinates, that is

$$u^2 + v^2 \leq w \leq 10.$$

Let S be this region in uvw-coordinates. We have

$$\int_R dV_{xyz} = \int_S \left|J_T\right| dA_{uvw}$$

where T is the transformation from uvw to xyz-coordinates. We have

$$\left|J_T\right| = \frac{1}{\left|J_{T^{-1}}\right|}$$

and $J_{T^{-1}}$ is the determinant of

$$\begin{pmatrix} 1 & 0 & -6 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

which is just $\frac{1}{2}$. Therefore $J_T = 2$. We have

$$\text{volume}(R) = 2\,\text{volume}(S).$$

Now we convert to cylindrical coordinates:

$$\begin{aligned}
\int_S dA_{uvw} &= \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{10}} \int_{w=r^2}^{10} r\, dw\, dr\, d\theta \\
&= 2\pi \int_{r=0}^{\sqrt{10}} rw\Big|_{w=r^2}^{10} dr \\
&= 2\pi \int_{r=0}^{\sqrt{10}} \left(10r - r^3\right) dr \\
&= 2\pi(5r^2 - \frac{r^4}{4})\Big|_{r=0}^{\sqrt{10}} \\
&= 2\pi(50 - 25) = 50\pi.
\end{aligned}$$

So our final answer is

$$\text{volume}(R) = 2\,\text{volume}(S) = 100\pi.$$

□

Problem 6. (16 points) Consider the curve C with equations and inequalities,

$$(x-2)^2 + z^2 - 1 = y = 0$$
$$x \geq 2$$
$$z \geq 0$$

(a) (6 points) Find the centroid of C.

Solution. There are many ways to do this. The slickest may be to use Pappus's theorem in reverse. The surface area of a hemisphere of radius 1 is $2\pi(E(x|C)-2)\text{length}(C) = 2\pi$. Therefore $E(x|C)-2 = \frac{2}{\pi}$ so $E(x|C) = 2 + \frac{2}{\pi}$. A similar argument shows that $E(z|C) = \frac{2}{\pi}$. Since C is entirely located on the xz-plane, $E(y|C) = 0$.

The centroid is therefore

$$(E(x|C), E(y|C), E(z|C)) = \left(2 + \frac{2}{\pi}, 0, \frac{2}{\pi}\right)$$

□

Let S be the surface obtained by rotating C around the z-axis.

(b) (2 points) Find the surface area of S.

Solution. Apply Pappus's theorem: $\text{area}(S) = 2\pi\, E(x|C)\, \text{length}(C) = 2\pi(2 + \frac{2}{\pi})\frac{\pi}{2} = 2\pi^2 + 2\pi$. □

(c) (8 points) Compute $\int_S \mathbf{F}\cdot\mathbf{n}\,dA$ where $\mathbf{F}$ is the vector field $(0,0,1)$ and S is given the orientation pointing away from the y-axis. (Hint: use Stokes's theorem or the divergence theorem.)

Solution. Notice that $\mathbf{F} = \operatorname{curl}(z,0,0)$. Therefore we can apply Stokes's theorem. The boundary of S consists of the two curves $D : x^2+y^2-9 = z = 0$ and $E : x^2+y^2-4 = z-1 = 0$. The first goes counterclockwise around the positive y-axis and the second goes clockwise.

Notice that D is also the boundary of the disc T where $x^2+y^2 \leq 9$ and $z = 0$, with the orientation $\mathbf{n}_T = (0,0,1)$. Likewise E is the boundary of the disc U where $x^2+y^2 \leq 4$ and $z = 1$, with the orientation $\mathbf{n}_U = (0,0,-1)$.

Applying Stokes's theorem again, we see that

$$\int_D \mathbf{F}\cdot d\mathbf{r} = \int_T \operatorname{curl}(\mathbf{F})\cdot\mathbf{n}_T\,dA_T = \int_T (0,0,1)\cdot(0,0,1)\,dA_T = \operatorname{area}(T)$$
$$\int_E \mathbf{F}\cdot d\mathbf{r} = \int_U \operatorname{curl}(\mathbf{F})\cdot\mathbf{n}_U\,dA_U = \int_T (0,0,1)\cdot(0,0,-1)\,dA_U = -\operatorname{area}(U).$$

Therefore the integral we want is

$$\begin{aligned}\int_S \mathbf{F}\cdot\mathbf{n}\,dA &= \int_D \mathbf{F}\cdot d\mathbf{r} + \int_D \mathbf{F}\cdot d\mathbf{r}\\ &= \int_T \operatorname{curl}(\mathbf{F})\cdot\mathbf{n}_T\,dA + \int_U \operatorname{curl}(\mathbf{F})\cdot\mathbf{n}_U\\ &= \operatorname{area}(T) - \operatorname{area}(U) && = 9\pi - 4\pi\\ &= 5\pi\end{aligned}$$

Since both T and U are discs. □

Solution. Another solution uses the divergence theorem. Consider the solid R obtained by rotating the region $0 \leq x \leq 2+\sqrt{1-z^2}$ around the z-axis. The boundary of this region consists of three pieces:

$$\partial R = S + T - U$$

where T is the disc given by the inequality $x^2+y^2 \leq 4$ and the equation $z = 1$ and U is the disc given by $x^2+y^2 \leq 9$ and $z = 0$ and both discs are given the orientation $(0,0,1)$. By the divergence theorem

$$\int_R \operatorname{div}(\mathbf{F}) = \int_{\partial R} \mathbf{F}\cdot\mathbf{n}\,dA = \int_S \mathbf{F}\cdot\mathbf{n}\,dA + \int_T \mathbf{F}\cdot\mathbf{n}\,dA - \int_U \mathbf{F}\cdot\mathbf{n}\,dA.$$

We have $\operatorname{div}(\mathbf{F}) = 0$ so rearranging, we get

$$\begin{aligned}\int_S \mathbf{F}\cdot\mathbf{n}\,dA &= \int_U \mathbf{F}\cdot\mathbf{n}\,dA - \int_T \mathbf{F}\cdot\mathbf{n}\,dA\\ &= \int_U dA - \int_T dA\\ &= \operatorname{area}(U) - \operatorname{area}(T)\\ &= 9\pi - 4\pi\\ &= 5\pi.\end{aligned}$$

□

Problem 7. (10 points) Let $\mathbf{F}$ be a vector field such that

$$\begin{aligned} \operatorname{div}(\mathbf{F}) &= 0 \\ \mathbf{F}(x,y,0) &= (\ ?\ ,\ ?\ , xy) \\ \mathbf{F}(x,0,z) &= (\ ?\ , xz,\ ?\ ) \\ \mathbf{F}(0,y,z) &= (yz,\ ?\ ,\ ?\ ). \end{aligned}$$

(The question marks stand for things that you will not need to complete this problem.) Suppose that S is the triangle with vertices $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$ with normal vector $\mathbf{n} = \frac{1}{\sqrt{3}}(1,1,1)$. Compute $\int_S \mathbf{F}\cdot\mathbf{n}\,dA$.

Solution. Consider the tetrahedron T defined by $x+y+z \le 1$ and $x \ge 0$, $y \ge 0$, and $z \ge 0$. The boundary ∂T is made up of S and three triangles:

$$\begin{array}{llllll} T_1: & x=0 & y+z\le 1 & y\ge 0 & z\ge 0 & \mathbf{n}=(-1,0,0) \\ T_2: & y=0 & x+z\le 1 & x\ge 0 & z\ge 0 & \mathbf{n}=(0,-1,0) \\ T_3: & z=0 & x+y\le 1 & x\ge 0 & y\ge 0 & \mathbf{n}=(0,0,-1). \end{array}$$

By the divergence theorem,

$$\int_S \mathbf{F}\cdot\mathbf{n}\,dA = -\int_{T_1} \mathbf{F}\cdot\mathbf{n}\,dA - \int_{T_2} \mathbf{F}\cdot\mathbf{n}\,dA - \int_{T_3} \mathbf{F}\cdot\mathbf{n}\,dA.$$

The three integrals on the right are the same by symmetry, so we only need to compute one of them.

$$\begin{aligned} \int_S \mathbf{F}\cdot\mathbf{n}\,dA &= -3\int_{T_1} \mathbf{F}\cdot\mathbf{n}\,dA \\ &= -3\int_{T_1} (yz,\ ?\ ,\ ?\ )\cdot(-1,0,0)\,dA \\ &= 3\int_{y=0}^{1}\int_{z=0}^{1-y} yz\,dz\,dy \\ &= 3\int_{y=0}^{1} \frac{y(1-y)^2}{2}\,dy \\ &= \frac{3}{2}\int_{y=0}^{1} \left((1-y)^2-(1-y)^3\right)dy \qquad \text{a calculational trick} \\ &= \frac{3}{2}\left(\frac{(1-y)^3}{3} - \frac{(1-y)^4}{4}\right)\Big|_{y=0}^{1} \\ &= \frac{3}{2}\left(\frac{1}{3}-\frac{1}{4}\right) \\ &= \frac{1}{8}. \end{aligned}$$

□

Problem 8. (8 points)

(a) (4 points) Find all values of a, b, c, and d such that the vector field

$$\mathbf{F} = \big(ax + by, cx + dy\big)$$

is conservative on the plane. **Justify your answer.**

Solution. On a simply connected region, a vector field is conservative if and only if its curl is zero. Since the plane is simply connected, we only have to check whether the curl is zero. The curl is $c - b$. So $\mathbf{F}$ will be a gradient of a vector field if and only if $b = c$. □

(b) (4 points) Suppose that a surface S is parameterized with coordinates u and v and

$$\frac{\partial(x,y)}{\partial(u,v)} = 2 \qquad \frac{\partial(z,x)}{\partial(u,v)} = 1 \qquad \frac{\partial(y,z)}{\partial(u,v)} = -2.$$

What is the surface area traced out by the parameters $-1 \le u \le 2$ and $-1 \le v \le 1$?

Solution. The Jacobian is the length of the vector $(2, 1, -2)$, which is $\sqrt{4+1+4} = 3$. Therefore the area is 3 times the area traced out in uv-coordinates, which is $3(2) = 6$. The surface area is therefore $3(6) = 18$. $\square$

Problem 9. (8 points) Let S be the paraboloid

$$z = x^2 + y^2 \leq 4.$$

Suppose that S is rotating around the axis $(1, 1, 1)$. Find the points of S that will not experience a Coriolis effect (recall that these are the points of S where a normal vector is perpendicular to the axis of rotation). Indicate these points on a sketch of S.

Solution. We find a formula for the normal vector. Parameterize the surface by

$$\begin{aligned} x &= u \\ y &= v \\ z &= u^2 + v^2 \\ u^2 + v^2 &\leq 4. \end{aligned}$$

The normal vector (with respect to this parameterization) is

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (1, 0, 2u) \times (0, 1, 2v) = (-2u, -2v, 1).$$

The Coriolis effect will not occur when $(-2u, -2v, 1)$ is perpendicular to $(1, 1, 1)$—that is, when $(-2u, -2v, 1) \cdot (1, 1, 1) = 0$. We obtain the equation,

$$-2u - 2v + 1 = 0.$$

That is, $u + v = \frac{1}{2}$.

Since $u = x$ and $v = y$, a point on the surface of the paraboloid will not experience a Coriolis effect if $x + y = \frac{1}{2}$. Here is a picture of those points:

□

Problem 10. (10 points) In order to discourage guessing, each of the multiple choice questions below is worth 2 points for a correct answer and -1 point for an incorrect answer. Problems that require justification have additional point values, as indicated.

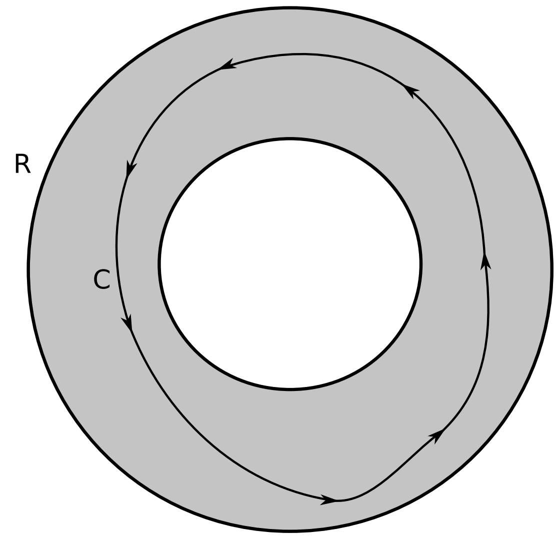
(a) (2 points) Let R be the shaded region to the right and suppose that $\int_C \mathbf{F} \cdot (dx, dy) = 0$. Decide whether

(i) there is a function f on R such that $F = \text{grad}(f)$,

(ii) such a function may exist but is not guaranteed, or

(iii) it is impossible that there is such a function.

Indicate your answer by circling the numeral of whichever response is true.



Solution. The answer is (ii). The vector field $(0,0)$ is consistent with the hypotheses, and this is certainly a gradient, so we can rule out (iii). On the other hand, it is possible that the curl of $\mathbf{F}$ could be non-zero, in which case having $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ doesn't tell us anything about whether F is the gradient of a function. For example, the vector field

$$\mathbf{F} = \left(\frac{-y}{x^2 + y^2} + y, \frac{x}{x^2 + y^2} - x \right)$$

gives the integral zero when integrated around the circle $x^2 + y^2 = 1$. However it is the gradient of any function on any region because its curl is nonzero. $\square$

(b) (2 points) There is a closed curve C inside the region R at right such that

$$\int_C \mathbf{F} \cdot (dx, dy) \neq 0.$$

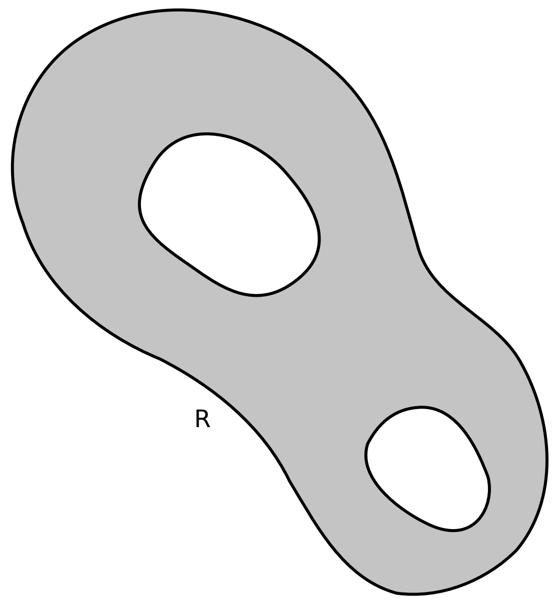
Decide whether

(i) $\operatorname{curl}(\mathbf{F}) = 0$,

(ii) it is possible that $\operatorname{curl}(\mathbf{F}) = 0$ but not guaranteed, or

(iii) it is impossible that $\operatorname{curl}(\mathbf{F}) = 0$.

Indicate your answer by circling the numeral of whichever response is true.



Solution. The answer is (ii). If the point $(0,0)$ were contained in one of the holes then the vector field $\mathbf{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ would have curl zero, but there would be a loop C such that $\int_C \mathbf{F} \cdot (dx, dy) \neq 0$. Therefore we can rule out (iii).

On the other hand, if C were a contractible loop inside of R then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \operatorname{curl}(F)\, dA \neq 0$ would tell us that $\operatorname{curl}(F) \neq 0$ somewhere on R. Therefore we can also rule out (i). The answer is (ii). $\square$

(c) (6 points) Assume that $\mathbf{F}$ is a vector field that is nice (its components have all partial derivatives of all orders) on the whole plane except at the points O, P, and Q, that $\operatorname{curl}(\mathbf{F}) = 0$ where it is defined, and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_D \mathbf{F} \cdot d\mathbf{r} = \int_E \mathbf{F} \cdot d\mathbf{r} = 0.$$

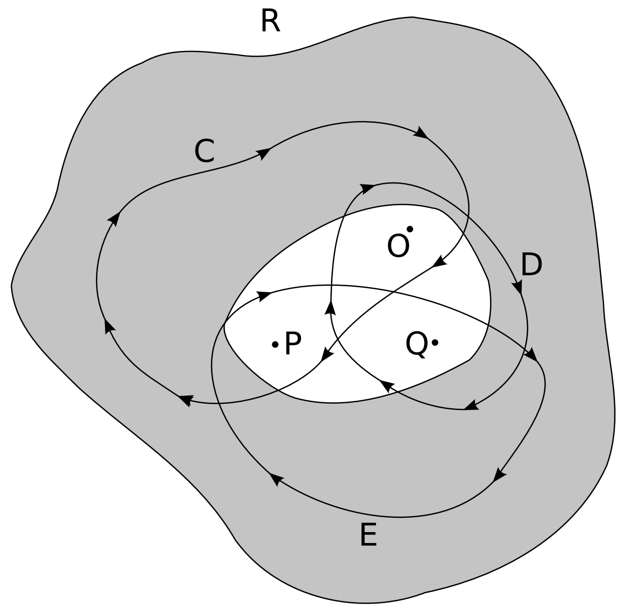
Decide whether

(i) on R, the vector field $\mathbf{F}$ is the gradient of a function,

(ii) $\mathbf{F}$ might or might not be the gradient of a function on R, or

(iii) $\mathbf{F}$ is not the gradient of a function on R.

Indicate your answer by circling the numeral of whichever response is true. **Then justify your answer below.**



Solution. First notice that because $\operatorname{curl}(\mathbf{F}) = 0$ on R, if a curve A can be continuously deformed within R to another curve B then $\int_A \mathbf{F} \cdot d\mathbf{r} = \int_B \mathbf{F} \cdot d\mathbf{r}$. This tells us that we can check if $\mathbf{F}$ is the conservative—which is equivalent to being the gradient of a function—by checking that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any loop that encloses the hole of R.

If we add the curves together, we get a curve $C + D + E$ that goes around each point a total of two times, clockwise. This is twice the loop the circles the hole clockwise, and $\int_{C+D+E} \mathbf{F} \cdot d\mathbf{r} = 0$. Therefore $\mathbf{F}$ is the gradient of a function: the answer is (i). $\square$

Problem 11. (extra credit: 10 points) Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = \Big(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\Big).$$

Define a new vector field

$$\mathbf{G}(x_0,y_0) = \int_R \mathbf{F}(x_0-x, y_0-y)\, dA_R = \int_R \mathbf{F}(x_0-x, y_0-y)\, dx\, dy$$

where R is the disc $x^2+y^2 \le 1$.

(a) (8 points) Compute $\int_C \mathbf{G} \cdot (dx, dy)$ where C is the loop $x^2+y^2=4$, oriented counterclockwise.

Solution. Write $\mathbf{r}_0 = (x_0, y_0)$ and $\mathbf{r} = (x,y)$. First we change the order of integration:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \Big(\int_R \mathbf{F}(x_0-x, y_0-y)\, dA\Big) \cdot d\mathbf{r}_0 \\ &= \int_R \Big(\int_C \mathbf{F}(x_0-x, y_0-y) \cdot d\mathbf{r}_0\Big) dA.\end{aligned}$$

Now, notice that as we do this integral, $\mathbf{r}$ ranges over points of R. Every one of these points is contained in the loop C, and we know that $\int_C \mathbf{F}(x_0-x, y_0-y) \cdot d\mathbf{r}_0 = -2\pi$ in this case. Therefore we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_R (-2\pi) dA = -2\pi \operatorname{area}(R) = -2\pi^2.$$

$\square$

(b) (2 points) Find a number a such that on the region S defined by the inequalities $4 \le x^2+y^2 \le 9$, the vector field $\mathbf{G} + a\mathbf{F}$ is the gradient of a function.

Solution. For any point $\mathbf{r}_0$ outside of R, we have

$$\operatorname{curl}(\mathbf{G}) = \int_R \operatorname{curl}(\mathbf{F}(x_0-x, y_0-y))\, dA = \int_R 0\, dA = 0.$$

Since $\operatorname{curl}(\mathbf{F}) = 0$ also, $\mathbf{G} + a\mathbf{F}$ will be conservative on S if and only if its integral around a loop encircling the hole is zero. The curve C from the last part is such a curve. We have

$$\int_C (\mathbf{G} + a\mathbf{F}) \cdot d\mathbf{r}_0 = -2\pi^2 + 2\pi a.$$

We can get this to be zero by making $a = \pi$. (Notice that this is the area of R.) $\square$

Extra space — do not detach.

Extra space — do not detach.