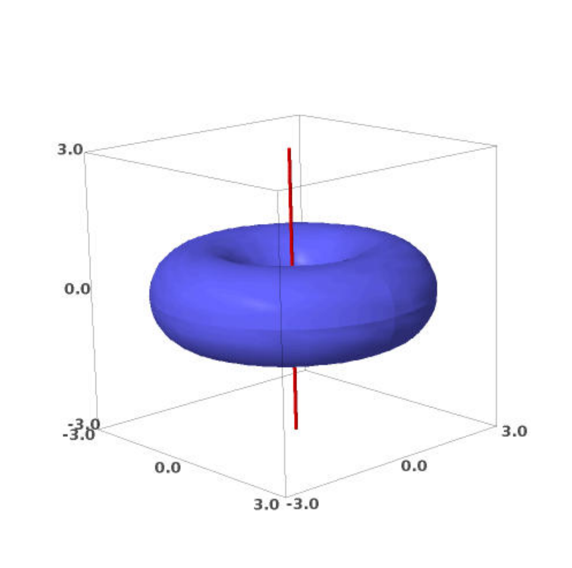
Math 52 – Spring 2012
Exam #2

You can use any resources you like—books, notes, internet, etc.—except other people. Note: talking to people on the internet (e.g., on question and answer sites) counts as talking to other people.

The exam is due by 5pm on Wednesday, May 23.

Problem 1. (11 points) The surface below is a torus that we will call S. The red line is the z-axis. You can see more visualizations of this torus at this webpage: http://nt.sagenb.org/home/pub/165/.



The following is a parameterization of this torus:

$$x = (2 + \cos(v))\cos(u) \qquad 0 \le u \le 2\pi$$
$$y = (2 + \cos(v))\sin(u) \qquad 0 \le v \le 2\pi$$
$$z = \sin(v)$$

(a) (2 points) Find a normal vector to the surface of S at the point with uv-coordinates (u, v) (it does not have to be a unit normal vector).

Solution. We have

$$\frac{\partial x}{\partial u} = -(2 + \cos(v))\sin(u) \qquad \frac{\partial y}{\partial u} = (2 + \cos(v))\cos(u) \qquad \frac{\partial z}{\partial u} = 0$$
$$\frac{\partial x}{\partial v} = -\sin(v)\cos(u) \qquad \frac{\partial y}{\partial v} = -\sin(v)\sin(u) \qquad \frac{\partial z}{\partial v} = \cos(v).$$

The cross product is

$$\mathbf{n} = \begin{pmatrix} (2+\cos(v))\cos(u)\cos(v) \\ (2+\cos(v))\sin(u)\cos(v) \\ (2+\cos(v))\sin(v)(\cos(u)^2+\sin(u)^2) \end{pmatrix}$$

$$= \begin{pmatrix} (2+\cos(v))\cos(u)\cos(v) \\ (2+\cos(v))\sin(u)\cos(v) \\ (2+\cos(v))\sin(v) \end{pmatrix}.$$

□

Solution. The following is notationally simpler:

$$\frac{\partial x}{\partial u} = -y \qquad \frac{\partial y}{\partial u} = x \qquad \frac{\partial z}{\partial u} = 0$$
$$\frac{\partial x}{\partial v} = \frac{xz}{r} \qquad \frac{\partial y}{\partial v} = -\frac{yz}{r} \qquad \frac{\partial z}{\partial v} = r-2$$

where $r = \sqrt{x^2+y^2} = 2+\cos(v)$.

The cross product is

$$\big(x(r-2), y(r-2), \frac{z(x^2+y^2)}{r}\big) = \big(x(r-2), y(r-2), zr\big).$$

□

(b) (2 points) Compute the Jacobian of the transformation from uv-coordinates to xyz-coordinates *on the torus.* In other words, compute the ratio between the infinitesimal area dA_S and dA_{uv} at a point with uv-coordinates (u, v).

Solution. We can compute the length of the cross product to get the ratio of infinitesimal surface area. We get

$$\left|\mathbf{n}\right|^2 = (2+\cos(v))^2$$

by using the identities $\cos(u)^2+\sin(u)^2 = 1$ and $\cos(v)^2+\sin(v)^2 = 1$. Thus, $\left|\mathbf{n}\right| = 2+\cos(v)$ (notice that this is always positive, so we don't have to take an absolute value). Thus

$$\left|J_T(u,v)\right| = 2+\cos(v)$$

if T is the transformation from uv-coordinates to xyz-coordinates on the torus. □

Solution. We can also compute like this:

$$\begin{aligned}\left|\mathbf{n}\right|^2 &= x^2(r-2)^2+y^2(r-2)^2+r^2z^2 \\ &= r^2(r-2)^2+r^2z^2 = r^2\end{aligned}$$

since $(r-2)^2+z^2 = 1$. Therefore $\left|J_T(u,v)\right| = r = 2+\cos(v)$. □

(c) (2 points) Find the surface area of S.

Solution. In uv-coordinates, the torus corresponds to the rectangle R where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. Now we get

$$\begin{aligned}\text{area}(S) &= \int_S dA_S = \int_R \left|J_T(u,v)\right| dA_R = \int_{u=0}^{2\pi}\int_{v=0}^{2\pi}(2+\cos(v))\,dv\,du \\ &= \int_{u=0}^{2\pi} 4\pi\,du = 8\pi^2.\end{aligned}$$

□

Solution. We can also compute by Pappus's theorem: the surface area is $2\pi\, E(f|T)\,\text{length}(T)$ where T is the circle $(x-2)^2 + y^2 \leq 1$ and $E(f|T)$ is the average distance of the center of that circle from the $x = 0$ axis. That distance is 2 and the circumference of the circle is 2π, so we get

$$\text{area}(S) = 8\pi^2.$$

□

(d) (1 point) Find the curl of the vector field $(-y, x, 0)$. (Note that this corresponds to rotation at a constant rate around the z-axis.)

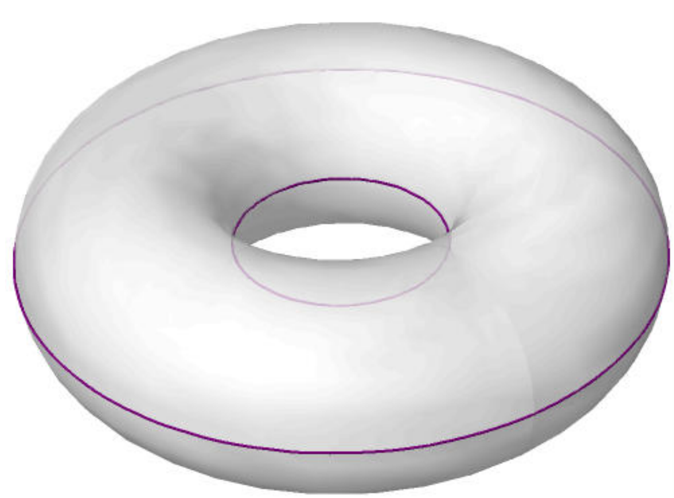
Solution.

$$\text{curl}(-y, x, 0) = (0, 0, 2)$$

□

(e) (2 points) Suppose that the torus is rotating in the velocity vector field $(-y, x, 0)$ of the previous part. On a picture of the torus, such as the one below (you may also draw your own if you prefer), indicate the places where a person standing on the surface of the torus would experience no Coriolis effect. (The red line is the z-axis and the yellow line is the x-axis.)

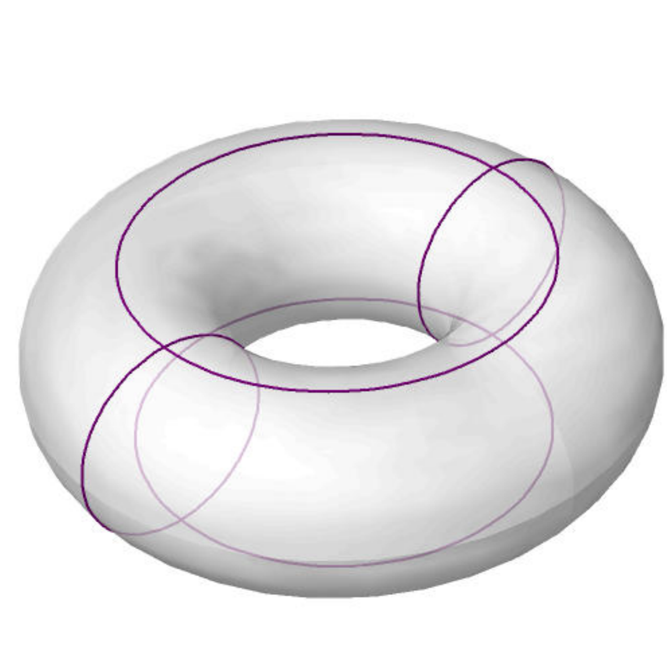
Solution. The normal vector to the surface of the torus at a point with coordinates (x, y, z) is $(x(r-2), y(r-2), zr)$. This will be perpendicular to $(0, 0, 2)$ when $zr = 0$, that is, when $z = 0$ or $r = 0$. There are no points on the torus with $r = 0$ so the only places with no Coriolis effect are where $z = 0$.



□

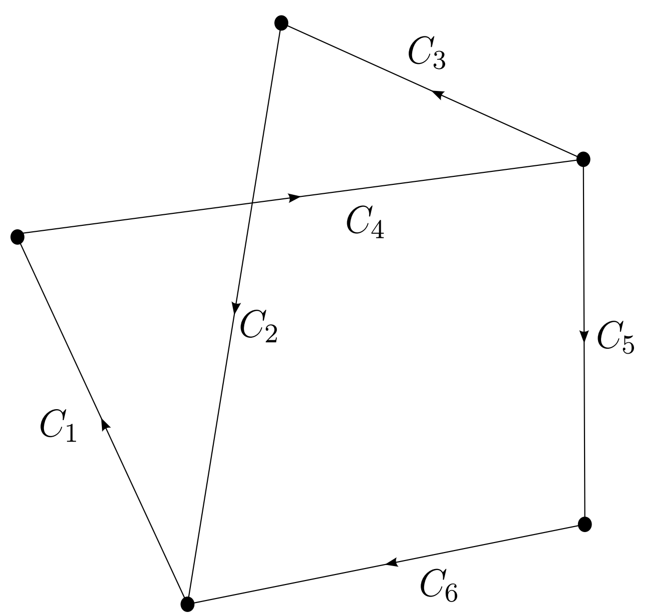
(f) (2 points) Now suppose the torus is rotating in the velocity vector field $(0, -z, y)$. Indicate the points on the surface of the torus where there is no Coriolis effect.

Solution. The curl of this vector field is $(2, 0, 0)$. The normal vector at (x, y, z) is $(x(r-2), y(r-2), zr)$. This is perpendicular to $(2, 0, 0)$ when $x(r-2) = 0$, i.e., when $x = 0$ or $r = 2$. These regions are depicted below.



□

Problem 2. (6 points) Suppose that F is a nice vector field on the entire plane. Consider curves $C_1, \dots, C_6$ in the following configuration.



Assume that

$$\int_{C_1} F \cdot (dx, dy) = 1 \qquad \int_{C_2} F \cdot (dx, dy) = 4 \qquad \int_{C_3} F \cdot (dx, dy) = 2$$

$$\int_{C_4} F \cdot (dx, dy) = 4 \qquad \int_{C_5} F \cdot (dx, dy) = 3 \qquad \int_{C_6} F \cdot (dx, dy) = 1.$$

(a) (2 points) Could F be a conservative vector field?

Solution. No: if F were conservative then $\int_{C_1+C_4+C_5+C_6} F \cdot (dx, dy)$ must be zero. But here it is $1+4+3+1 = 9 > 0$. $\square$

(b) (2 points) Is it possible that $\text{curl}(F)$ is ≥ 0 at *all* points of the plane?

Solution. No: the integral around the counterclockwise loop $-C_1 - C_4 - C_6 - C_5$ is $-9 < 0$ but by Green's theorem this would be positive if $\text{curl}(F)$ were ≥ 0 everywhere: if R is a region with boundary $-C_1 - C_4 - C_6 - C_5$ then $\int_R \text{curl}(F)\, dA = -9$; this is impossible if $\text{curl}(F) \geq 0$ always.

Notice that it is very important to select a loop that is *counterclockwise* and produces a negative integral. In the statement of Green's theorem the boundary of a plane region has the counterclockwise orientation; a clockwise loop that gives a negative integral is the same as a counterclockwise loop with a positive integral and this is perfectly consistent with $\text{curl}(F) \geq 0$. $\square$

(c) (2 points) Is it possible that $\text{curl}(F)$ is $\neq 0$ at *all* points of the plane?

Solution. No: the integral around the counterclockwise loop $C_3 + C_2 - C_6 - C_5$ is $4+2-3-1 = 2 > 0$ and the integral around the counterclockwise loop $-C_1 - C_4 - C_5 - C_6$ is $-9 < 0$. Therefore the curl is negative at some points and positive at others. But it is also continuous because F is nice, so it must be zero somewhere. $\square$

Problem 3. (8 points) Consider the region R defined by the inequalities,

$$0 \leq 4x + 3y \leq 25$$
$$0 \leq 4y - 3x \leq 25.$$

Suppose this region has density $\delta(x, y) = x^2 + y^2$.

In this problem you will have to write down various single variable integrals. Your score will be based not only on the correctness of your answer, but *also* on the number of single variable integrals you use (i.e., the number of integral signs in your formula): the fewer the better. (This is intended to discourage you from solving this problem by brute force.)

(a) (5 points) Compute the mass of R.

Solution. Make the change of coordinates $u = 4x + 3y$ and $v = 4y - 3x$. Then the region becomes $0 \leq u \leq 25$ and $0 \leq v \leq 25$. Let S be this region.

If T is the transformation from uv-coordinates to xy-coordinates then $T^{-1}(x, y) = (4x + 3y, 4y - 3x)$. Therefore

$$J_{T^{-1}} = \det \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix} = 25.$$

Therefore

$$J_T = \frac{1}{25}.$$

We can compute T from T^{-1} by linear algebra. We get

$$T(u, v) = \frac{1}{25} \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

That is,

$$T(u, v) = \Big(\frac{4u - 3v}{25}, \frac{3u + 4v}{25}\Big).$$

We can now evaluate

$$\delta(T(u, v)) = \Big(\frac{4u - 3v}{25}\Big)^2 + \Big(\frac{3u + 4v}{25}\Big)^2 = \frac{u^2 + v^2}{25}.$$

Now we can compute

$$\begin{aligned}
\int_R \delta(x,y)\,dA_R &= \int_S \big|J_T(u,v)\big|\delta(T(u,v))dA_S \\
&= \int_S \frac{u^2+v^2}{25^2} dA_S \\
&= \frac{1}{25^2}\int_{u=0}^{25}\int_{v=0}^{25}(u^2+v^2)\,dv\,du \\
&= \frac{1}{25^2}\int_{u=0}^{25}\left(25u^2+\frac{25^3}{3}\right)du \\
&= \frac{1}{25^2}\left(\frac{25^4}{3}+\frac{25^4}{3}\right) \\
&= \frac{2(25^2)}{3} = \frac{1250}{3}.
\end{aligned}$$

□

(b) (3 points) Write down an expression using single variable integrals (iterated integrals are okay, but multivariable integrals are not) for $E(f|R)$, where $f(x,y)$ is the distance of the point (x,y) from the origin.

Solution. Since $f(x,y) = \sqrt{x^2+y^2} = \sqrt{\delta(x,y)}$,

$$f(T(u,v)) = \frac{1}{5}\sqrt{u^2+v^2}.$$

The integral is then

$$\begin{aligned}
\int_S \frac{f(T(u,v))\delta(T(u,v))}{25}\,dA_S &= \int_{u=0}^{25}\int_{v=0}^{25}\frac{u^2+v^2}{25}\frac{\sqrt{u^2+v^2}}{5}\frac{1}{25}\,dv\,du \\
&= \frac{1}{5^5}\int_{u=0}^{25}\int_{v=0}^{25}(u^2+v^2)^{3/2}\,dv\,du
\end{aligned}$$

so we conclude that

$$E(f|R) = \frac{\int_R f\,dA_R}{\mathrm{mass}(R)} = \frac{\int_{u=0}^{25}\int_{v=0}^{25}(u^2+v^2)^{3/2}\,du\,dv}{\frac{1250}{3}} = \frac{3}{2(5^9)}\int_{u=0}^{25}\int_{v=0}^{25}(u^2+v^2)^{3/2}\,du\,dv.$$

This comes out neater with a change of coorinates $u_1 = \frac{u}{25}$ and $v_1 = \frac{v}{25}$:

$$E(f|R) = \frac{15}{2}\int_{u_1=0}^{1}\int_{v_1=0}^{1}(u_1^2+v_1^2)^{3/2}\,du_1\,dv_1.$$

□

Problem 4. (8 points) Suppose that a circle of radius 1 is *rolling* from left to right inside the plane so that the position of its center at time t is $(t,0)$. Assume that a point on the edge of the circle has been marked—call this point P. You may want to imagine P as a piece of gum stuck to a rolling bicycle wheel. The curve traced out by C is called a *cycloid*.

Assume that when $t=0$, the position of P is $(0,1)$.

(a) (1 point) After $t=0$, the y-coordinate of P next returns to 1 at $t=2\pi$. Explain why this is.

Solution. One full rotation of the wheel: $t=2\pi$. □

(b) (1 point) Let C be the curve traversed by P between $t = 0$ and the value you found in the last part. Find a parameterization of C. Do not forget to indicate the values of t where your parameterization begins and ends.

Solution.

$$\begin{aligned} x(t) &= t + \sin(t) \\ y(t) &= \cos(t) \\ 0 \leq t &\leq 2\pi \end{aligned}$$

□

(c) (2 points) Compute the distance travelled by P between $t = 0$ and the value you computed in the first part. (Hint: $1 + \cos(t) = 2\cos\left(\frac{t}{2}\right)^2$.)

Solution. We have $dx = (1 + \cos(t))dt$ and $dy = -\sin(t)dt$. Therefore

$$\begin{aligned} (dx)^2 + (dy)^2 &= (1 + 2\cos(t) + \cos(t)^2 + \sin(t)^2)(dt)^2 \\ &= (2 + 2\cos(t))(dt)^2 \\ &= 4\cos(t/2)^2(dt)^2 \end{aligned}$$

so we can compute

$$\begin{aligned} \text{length}(C) &= \int_C dL_C \\ &= \int_{t=0}^{2\pi} \sqrt{(dx)^2 + (dy)^2} \\ &= \int_{t=0}^{2\pi} \sqrt{4\cos(t/2)^2}\, dt \\ &= \int_{t=0}^{2\pi} 2\left|\cos(t/2)\right| dt \\ &= 4\int_{t=0}^{\pi} \cos(t/2)\, dt && \text{since } \int_{t=0}^{\pi} \left|\cos(t/2)\right| dt = \int_{t=\pi}^{2\pi} \left|\cos(t/2)\right| dt \\ & && \text{and } \cos(t/2) \geq 0 \text{ when } 0 \leq t \leq \pi \\ &= 8\sin(t/2)\Big|_{t=0}^{\pi/2} = 8. \end{aligned}$$

□

(d) (1 point) Suppose that a force field $F = (x, 0)$ is acting on the wheel. Compute the work done by this field as P traverses the path from $t = 0$ to the value you computed in the first part. (Hint: is F conservative?)

Solution. $F = \text{grad}(\frac{1}{2}x^2)$. We can evaluate this at the endpoints:

$$\frac{1}{2}(2\pi)^2 - \frac{1}{2}(0)^2 = 2\pi^2.$$

□

(e) (1 point) Let F be the vector field $(y, 0)$. Compute $\int_C F \cdot (dx, dy)$.

Solution. This one isn't conservative so we have to do some calculation. We have $dx = (1 + \cos(t))\, dt$ and $y = \cos(t)\, dt$

$$\int_C y\, dx = \int_{t=0}^{2\pi} \cos(t)(1 + \cos(t))\, dt = \int_{t=0}^{2\pi} \cos(t)^2\, dt = \pi.$$

$\square$

(f) (2 point) Use your answer from the last part to compute the area above C and below the line $y = 1$.

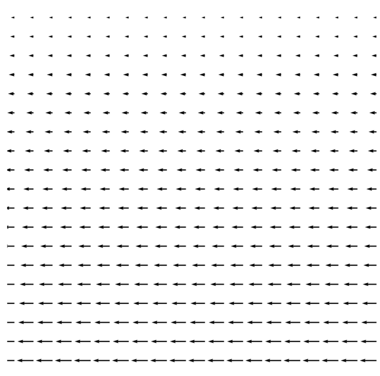
Solution. By Green's theorem we can get the area by integrating $-y\, dx$ around the boundary. We already computed $\int_C y\, dx = \pi$ so $\int_C (-y dx) = -\pi$. We can find the integral on the line connecting $(2\pi, 1)$ to $(0, 1)$ without even calculating: it is 2π. (Note the choice of orientation on this line coming from taking the boundary of the region.) Therefore the area is $2\pi - \pi = \pi$. $\square$

Problem 5. (4 points) Determine which of the following vector fields are gradients of functions on the region R that is indicated.

(a) (1 point) $F = (2x - 3y, 3x - y)$ and R is the whole plane.

Solution. $\mathrm{curl}(F) = 3 + 3 \neq 0$ so F can't be the gradient of anything. $\square$

(b) (1 point) F is the vector field displayed below and R is the square region of the plane on which the vector field is shown.



Solution. Observe that the field is always directed horizontally, but only varies in the vertical direction. Therefore the vector field is of the form $F = (a, 0)$ where $\frac{\partial a}{\partial y} \neq 0$. Thus $\mathrm{curl}(F) \neq 0$ so the vector field is not a gradient.

We can also see this by drawing a closed curve on the region. $\square$

(c) (2 points) $F = \left(\frac{-y}{(x-1)^2 + y^2}, \frac{x-1}{(x-1)^2 + y^2}\right) - \left(\frac{-y}{(x+1)^2 + y^2}, \frac{x+1}{(x+1)^2 + y^2}\right)$;

R is the annulus $4 \leq x^2 + y^2 \leq 9$.

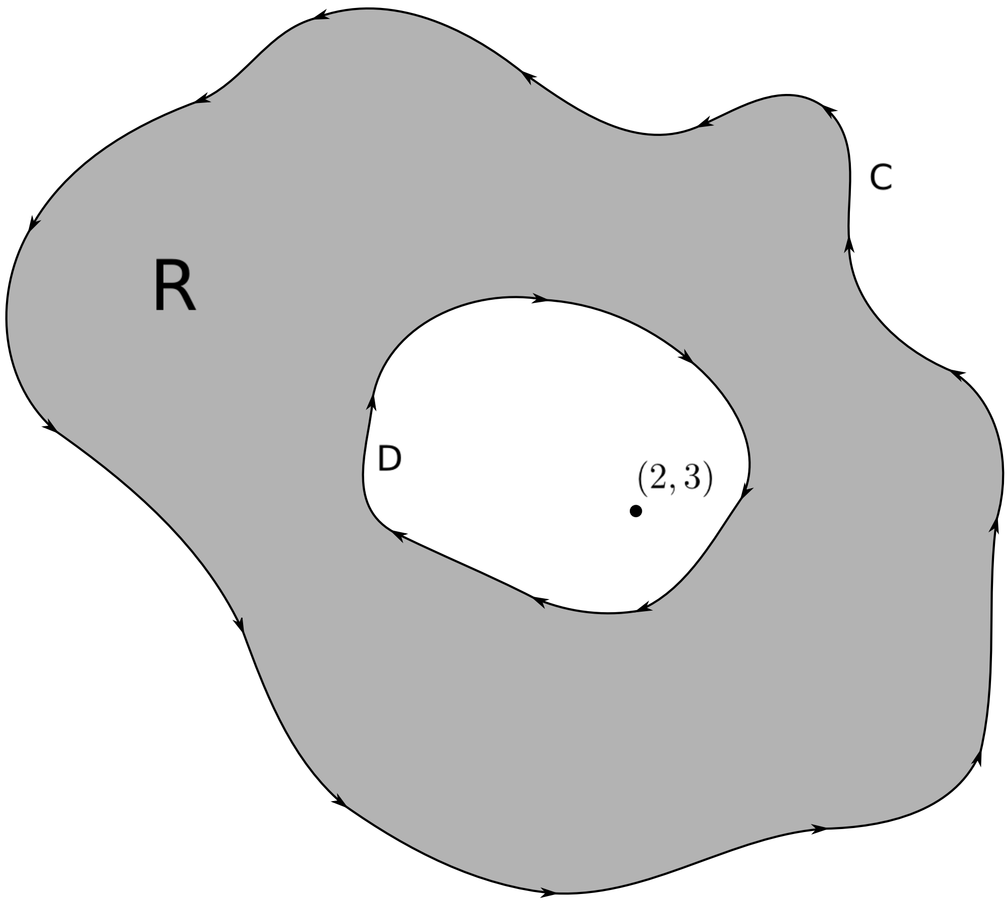
Solution. Write $F_1 = \left(\frac{-y}{(x-1)^2+y^2}, \frac{x-1}{(x-1)^2+y^2}\right)$ and $F_2 = \left(\frac{-y}{(x+1)^2+y^2}, \frac{x+1}{(x+1)^2+y^2}\right)$. We know that for any loop C, we have

$$\int_C F_1 \cdot d\mathbf{r} = \Big(\text{number of times } C \text{ circles } (1, 0)\Big)$$

$$\int_C F_2 \cdot d\mathbf{r} = (\text{number of times } C \text{ circles } (-1, 0).$$

Since any loop inside R must circle both points the same number of times, $\int_C (F_1 - F_2) \cdot d\mathbf{r} = 0$ for any closed curve inside C. Therefore F is conservative so it is the gradient of a function. $\square$

Problem 6. (7 points) Let R be the region below.



(a) (3 points) Construct a vector field F on R and a closed curve E inside R such that $\mathrm{curl}(F) = 0$ and

$$\int_E F \cdot (dx, dy) \neq 0.$$

Indicate the curve E by drawing it on the region.

Solution. Let

$$F = \Big(\frac{3-y}{(x-2)^2+(y-3)^2}, \frac{x-2}{(x-2)^2+(y-3)^2}\Big).$$

and let $E = C$. Then

$$\int_E F \cdot (dx, dy) = 2\pi.$$

□

(b) (1 point) Using Green's theorem, conclude that R is not simply connected.

Solution. Green's theorem tells us that if R is simply connected then any nice vector field with zero curl is conservative. But we have just seen that the vector field F has curl zero but is not conservative on R: there is a closed curve E in R but $\int_E F \cdot (dx, dy) \neq 0$ □

The boundary of R consists of the curves $C + D$, where C and D are as indicated in the image. The orientations on C and D are chosen so that R is always to the *left* as one traverses the curve in the indicated direction.

(c) (2 points) Demonstrate that if F is a vector field that is nice on R then

$$\int_R \mathrm{curl}(F)\, dA_R = \int_{\partial R} F \cdot (dx, dy) \tag{$*$}$$

where $\partial R = C + D$ with the orientations as indicated. Notice that R is not simply connected, so you can't apply Green's theorem (in the form we discussed in class) directly! (Hint: divide R into pieces where you can use Green's theorem in the form we discussed.)

Solution. Split R into two simply connected regions R_1 and R_2 so that $R = R_1 + R_2$ and $\partial R = \partial R_1 + \partial R_2$. Green's theorem applied to each piece gives:

$$\int_{R_1} \mathrm{curl}(F)\, dA = \int_{\partial R_1} F \cdot (dx, dy)$$
$$\int_{R_2} \mathrm{curl}(F)\, dA = \int_{\partial R_2} F \cdot (dx, dy)$$

Add the two lines together to get

$$\int_R \mathrm{curl}(F)\, dA = \int_{\partial R} F \cdot (dx, dy).$$

□

If R is any bounded region (possibly possessing holes) its boundary will be made up of multiple closed curves $C_1, \ldots, C_n$. We define the oriented boundary of R to be the sum $C_1 + \cdots + C_n$, with all curves oriented so that R is *to the left* of each curve when one traverses the curve according to its orientation.

(d) (1 point) Explain how you would use the ideas from this problem to show that Equation ($*$) is true for *any* region R.

Solution. We can divide any region into simply connected regions on which Green's theorem is already known to apply. Summing the contributions of the various pieces together, we get the desired result. □

Problem 7. (extra credit: 3 points) Suppose that R is a region (that may or may not be simply connected) and F and G are vector fields on R with the following properties:

(i) both F and G are nice on R,

(ii) $\mathrm{curl}(F) = \mathrm{curl}(G) = 0$, and

(iii) every vector field on R that has curl zero is equal to $aF + bG + \mathrm{grad}(f)$ for some numbers a and b and some function f that is nice on R.

Justify your answer to the following question:

(a) What are the largest and smallest numbers of holes the region R could have?

Solution. Recall that de Rham's theorem says that the maximal number of vector fields of curl zero that are linearly independent of the vector fields that are gradients is equal to the number of holes. Item (iii) above tells us that this number is between 0 and 2 so the number of holds must be between 0 and 2.

Smallest: 0.

Largest: 2.

□

Each of the following parts of this problem describes one or more *additional* assumptions about R, F, and G. In each situation, give the largest and smallest number of holes the region R could have, making sure to justify your answer.

(b) (iv) There is a closed curve C inside of R such that $\int_C (F + 3G) \cdot (dx, dy) \neq 0$.

Solution. The only way there could be zero holds is if both F and G were conservative. But then $\int_C (F + 3G) \cdot d\mathbf{r} = \int_C F \cdot d\mathbf{r} + 3 \int_C G d\mathbf{r}$ would be zero. Therefore we can rule out the possibility that there are zero holes.

In other words, there is at least one vector field of curl zero that is independent of the gradients.

Smallest: 1.

Largest: 2. □

(c) (iv) There is a closed curve C inside of R such that $\int_C F \cdot (dx, dy) \neq 0$, and
(v) there is a closed curve D inside of R such that $\int_D G \cdot (dx, dy) \neq 0$.

Solution. Either of these conditions guarantees that there is at least one hole, as in the last part. It might be the case that F and G measure the number of loops around two different holes of R, but we have no way of knowing this; it is perfectly possible that F and G could be the same, and that C and D could also be the same. Therefore there could be either 1 or 2 holes.

In other words, these conditions only guarantee one vector field of curl zero that is independent of the gradients.

Smallest: 1.

Largest: 2. □

(d) (iv) For every closed curve C inside of R we have $\int_C (F + 3G) \cdot (dx, dy) = 0$.

Solution. This rules out the possibility of two holes: $F + 3G$ must be the gradient of a function. Therefore every vector field on R that has curl zero can be expressed as $bG + \text{grad}(f)$ for some function f. That is, there is at most one vector field of curl zero that is linearly independent of the gradients. By de Rham's theorem there can therefore be 0 or 1 holes in R.

Smallest: 0.

Largest: 1. □

(e) (iv) There is a closed curve C inside of R such that $\int_C F \cdot (dx, dy) = 1$ and $\int_C G \cdot (dx, dy) = 2$, and
(v) there is a closed curve D inside R such that $\int_D F \cdot (dx, dy) = 1$ and $\int_E G \cdot (dx, dy) = 3$.

Solution. If there were fewer than 2 holes then F would be a multiple of G plus the gradient of a function: $F = bG + \text{grad}(f)$ for some number b and nice function f. But then

$$\int_E F \cdot d\mathbf{r} = \int_E (bG + \text{grad}(f)) \cdot \mathbf{r} = b \int_E G \cdot \mathbf{r}$$

for every loop E inside R. But the two equations

$$\int_C F \cdot d\mathbf{r} = \frac{1}{2} \int_C G \cdot d\mathbf{r} \int_C F \cdot d\mathbf{r} = \frac{1}{3} \int_C G \cdot d\mathbf{r}$$

tell us that b would have to be both $\frac{1}{2}$ and $\frac{1}{3}$, which is impossible.

Smallest: 2.

Largest: 2. □