Math 52 – Spring 2012

Exam #1

First name:

Last name:

Student number:

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Problem 1. *Let R be the region in the plane defined by the inequalities $|x - y - 2| \leq 2$ and $|x + y - 3| \leq 3$. (Hint: this entire problem can be done without computing an integral.)*

(a) Draw a sketch of the region R. Label the boundary curves of R and the points where they intersect.

(b) Compute the area of R.

(c) Find the centroid of R.

(d) Compute $\int_R x \, dA$.

Problem 2. *Let R be a solid sphere of radius 2 with density function δ (which is always ≥ 0). Assume that R is made up of three parts as follows:*

U *is defined by* $0 \leq \phi \leq \frac{\pi}{3}$ *and* $\text{mass}(U) = \text{volume}(U)$,

V *is defined by* $\frac{\pi}{3} \leq \phi \leq \pi$ *and* $0 \leq \rho \leq 1$ *and* $\text{mass}(V) = \text{volume}(V)$, *and*

W *is defined by* $\frac{\pi}{3} \leq \phi \leq \pi$ *and* $1 \leq \rho \leq 2$ *and* $\text{mass}(W) = \text{volume}(W)$.

(a) Find the volume of each of the regions U, V, and W.

(b) Find the largest possible value of $\int_R z\delta\, dV$, given the information above.

Problem 3. *Compute the volume of the 3-dimensional region defined by the inequalities*

$$2x - 2 \leq z \leq 8 - (x-1)^2 - y^2.$$

Problem 4. *Suppose that two real numbers are chosen randomly with the probability density*

$$\delta(x) = e^{-x^2}.$$

What is the expected absolute value of the difference between them? You may use the following facts without justification:

$$\textstyle\int_{x=-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \text{ and } -\frac{1}{2}e^{-x^2} \text{ is an antiderivative of } xe^{-x^2}\,dx.$$

Formula sheet

Coordinate systems:

$$\begin{aligned}
\text{Spherical} \to \text{Cartesian} &: (x, y, z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \\
\text{Spherical} \to \text{Cylindrical} &: (r, \theta, z) = (\rho\sin\phi, \theta, \rho\cos\phi) \\
\text{Cylindrical} \to \text{Cartesian} &: (x, y, z) = (r\cos\theta, r\sin\theta, z) \\
\text{Cartesian} \to \text{Spherical} &: (\rho, \phi, \theta) = (\sqrt{x^2 + y^2 + z^2}, \arctan\left(\frac{r}{z}\right), \arctan\left(\frac{y}{x}\right)) \\
\rho^2 &= x^2 + y^2 + z^2 = r^2 + z^2 \\
r^2 &= x^2 + y^2 \\
\mathrm{d}V &= \rho^2 \sin\phi \,\mathrm{d}\phi \,\mathrm{d}\theta \,\mathrm{d}\rho = r \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z
\end{aligned}$$

Pappus's Theorem (2D):

$$\mathrm{area}(Q) = 2\pi\, E(\text{distance from axis}|R)\,\mathrm{length}(R)$$

The centroid is $(E(X|R), E(Y|R)), E(Z|R))$.

$$E(x|R) = \frac{\int_R x\delta \,\mathrm{d}A}{\int_R \delta \,\mathrm{d}A} \qquad E(y|R) = \frac{\int_R y\delta \,\mathrm{d}A}{\int_R \delta \,\mathrm{d}A} \qquad E(z|R) = \frac{\int_R z\delta \,\mathrm{d}A}{\int_R \delta \,\mathrm{d}A)}$$

Inequalities $a^2 \le b^2 \iff |a| \le |b| \qquad |a| \le c \iff -c \le a \le c$

Trig Identities:

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin\theta = 2\cos\theta - 1$$

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta \qquad \csc^2\theta = 1 + \cot^2\theta$$

Half Angle Formulas: $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \qquad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

Expectation and probability

$$E(f|R) = \frac{\int_R f\delta \,dA}{\int_R \delta \,dA} = \frac{\int_R f\delta \,dA}{\mathrm{mass}(R)} \qquad P(S|R) = \frac{\int_S \delta \,dA}{\int_R \delta \,dA} = \frac{\mathrm{mass}(S)}{\mathrm{mass}(R)}$$

Integration by Parts $\displaystyle\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$

FIRST THEOREM OF PAPPUS : Volume of Revolution Suppose that a plane region R is revolved around an axis in its plane, generating a solid of revolution with volume V. Assume that R lies entirely on one side of the axis. The $V = Ad$ where $A = \mathrm{area}(R)$ and d is the distance travelled by the centroid.

SECOND THEOREM OF PAPPUS: Surface Area of Revolution Let the plane curve C be revolved around an axis in its plane that does not intersect the curve (except possibly in its endpoints). Then the area of the surface of revolution generated is $A = sd$ where s is the length of C and d is the distance travelled by the centroid.

$$\int \sin(x)\, dx = -\cos(x) + C \qquad \int \cos(x)\, dx = \sin(x)$$

Volume of sphere: $\dfrac{4\pi r^3}{3}$

Volume of cone: $\dfrac{Bh}{3}$ where $B = $ area of base