Math 3140 — Fall 2012
Assignment #10

Due Friday, November 30. Cite any sources you use.

Exercise 1. Let A and B be rings and $\varphi : A \to B$ a ring homomorphism. Prove the following statements:

(a) $\varphi(0) = 0$

(b) (extra credit) Give an example of two commutative rings A and B and a function $f : A \to B$ such that $f(x+y) = f(x)+f(y)$ and $f(xy)+f(x)f(y)$ for all $x, y \in A$ but $f(1) \neq 1$.[1] $\leftarrow_1$

(c) $\varphi(-x) = -\varphi(x)$ for all $x \in A$

(d) if x is a unit of A then $\varphi(x^{-1}) = \varphi(x)^{-1}$

Exercise 2. [Fra, §23, #26] Let A be a commutative ring. Define $A[x]$ to be the set of **polynomials** with coefficients in A. These are symbols

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

such that there exists an index n with $a_k = 0$ for $k > n$. This n is called the **degree** of the polynomial. If

$$\begin{aligned} f &= a_0 + a_1 x + a_2 x^2 + \cdots \\ g &= b_0 + b_1 x + b_2 x^2 + \cdots \end{aligned}$$

then define

$$\begin{aligned} f + g &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots \\ fg &= c_0 + c_1 x + c_2 x^2 + \cdots \end{aligned}$$

where

$$c_n = \sum_{m=0}^{n} a_m b_{n-m}.$$

Prove the following statements carefully:

(a) the product of two polynomials is a polynomial,

(b) multiplication of polynomials is commutative, and

(c) multiplication and addition of polynomials satisfy the distributive law.

[1] the problem that was here originally was incorrect!

In fact, $A[x]$ forms a commutative ring, though you do not have to check all of the details. (You may find a proof of the associativity of multiplication in [Fra, Theorem 22.2].)

Exercise 3. Let A be a ring.

(a) Show that there is **exactly one** ring homomorphism $\varphi : \mathbf{Z} \to A$.

(b) Show that there is **at most one** ring homomorphism $\varphi : \mathbf{Q} \to A$.

Exercise 4. Show that $\mathbf{R} \times \mathbf{R}$ is **not isomorphic** to $\mathbf{C}$ as a ring. (Hint: look for solutions to the equation $xy = 0$ in each of these rings.)

Exercise 5. (a) Let F be a field. Show that the only solution to $x^2 = 0$ in F is $x = 0$.

(b) Let A be the ring of symbols $a + b\epsilon$ with

$$(a + b\epsilon) + (a' + b'\epsilon) = (a + a') + (b + b')\epsilon$$
$$(a + b\epsilon)(a' + b'\epsilon) = aa' + (ab' + a'b)\epsilon.$$

Show that there is no injective homomorphism from A into a field.

References

[Fra] John B. Fraleigh. *A first course in abstract algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., seventh edition edition, 2002. ISBN-10: 0201763907, ISBN-13: 978-0201763904.