Math 3140 — Fall 2012
Assignment #9

Fri., Nov. 16. Remember to cite your sources. Do 6 of the following problems.

Exercise 1. Let $\widetilde{\varphi} : G \to G'$ be a homomorphism with $\ker(\widetilde{\varphi}) = H$.[1] ←1

(a) Prove that H is normal.

(b) Prove that the homomorphism $\varphi : G/H \to G'$ defined by $\varphi(gH) = \widetilde{\varphi}(g)$ is injective.

(c) Demonstrate that if $\widetilde{\varphi}$ is surjective then φ is an isomorphism.

Exercise 2. Prove that $\mathbf{C}^*/\{\pm 1\}$ is isomorphic to $\mathbf{C}^*$.

Exercise 3. (a) Prove that $\langle \rho^3 \rangle$ is normal in D_{12}.

(b) Prove that $D_{12}/\langle \rho^3 \rangle \cong D_3$.

Exercise 4. Find a product of cyclic groups that is isomorphic to $\mathbf{Z}^2/(4,6)\mathbf{Z}$. Prove that your answer is correct.

Exercise 5. Let $U = \{z \in \mathbf{C} \mid |z| = 1\}$. Show that there is an isomorphism

$$\varphi : \mathbf{R}/\mathbf{Z} \xrightarrow{\sim} U.$$

defined by $\varphi(t + \mathbf{Z}) = \cos(2\pi t) + i \sin(2\pi t)$. (Hint: use the angle sum identities or Euler's formula.)

Exercise 6. Let H be a subgroup of a group G. Construct a bijection between G/H and $H\backslash G$. (Hint: prove that the function $\varphi(gH) = Hg^{-1}$ defines a bijection from G/H to $H\backslash G$.) Conclude that if either one is finite then they have the same number of elements. This number is called the **index** of H in G and is denoted $(G : H)$.

Exercise 7. Prove that if H is a subgroup of G and $(G : H) = 2$ then H is normal in G.

Exercise 8. Let X be the set of unordered partitions of $\{1,2,3,4\}$ into 2-element subsets. An element of X is a set $\{S,T\}$ where $S,T \subset \{1,2,3,4\}$ and $S \cap T = \varnothing$ and $S \cup T = \{1,2,3,4\}$ and S and T each have 2 elements. In other words, an element of X is a set $\{\{a,b\},\{c,d\}\}$ where $\{a,b,c,d\} = \{1,2,3,4\}$.

(a) How many elements does X have? List them. Be careful to remember that $\{S,T\} = \{T,S\}$.

(b) Let S_4 act on X by the rule $g.\{S,T\} = \{gS, gT\}$ where $g\{a,b\} = \{g(a), g(b)\}$. For example,

$$(123)\big\{\{1,2\},\{3,4\}\big\} = \big\{\{2,3\},\{1,4\}\big\}.$$

Show that this is an action of S_4 on X.

(c) For each element of X, determine its stabilizer in S_4.

(d) Prove that $S_4/\{e,(12)(34),(13)(24),(14)(23)\} \cong S_3$.

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[1] correction: forgotten tilde; thanks Tyler