Math 3140 — Fall 2012
Assignment #7

Due Mon., Oct. 29. Remember to cite your sources.

Exercise 1. List the cosets of each of the following subgroups:

(a) $8\mathbf{Z} \subset \mathbf{Z}$

Solution.

$$8\mathbf{Z}, 1+8\mathbf{Z}, 2+8\mathbf{Z}, 3+8\mathbf{Z}, 4+8\mathbf{Z}, 5+8\mathbf{Z}, 6+8\mathbf{Z}, 7+8\mathbf{Z}$$

□

(b) $9\mathbf{Z}/36\mathbf{Z} \subset \mathbf{Z}/36\mathbf{Z}$

Solution.

$$9\mathbf{Z}/36\mathbf{Z}, 1+9\mathbf{Z}/36\mathbf{Z}, 2+9\mathbf{Z}/36\mathbf{Z}, 3+9\mathbf{Z}/36\mathbf{Z}, 4+9\mathbf{Z}/36\mathbf{Z},$$
$$5+9\mathbf{Z}/36\mathbf{Z}, 6+9\mathbf{Z}/36\mathbf{Z}, 7+9\mathbf{Z}/36\mathbf{Z}, 8+9\mathbf{Z}/36\mathbf{Z}$$

□

(c) $\langle\rho^3\rangle \subset D_6$ (where ρ is rotation by $60^\circ = \frac{\pi}{3}$)

Solution. Every element of D_6 can be written as $\rho^n\tau$ where τ is a reflection and $n \in \{0,1,2,3,4,5\}$. Let $H = \langle\rho^3\rangle$. The left and right cosets of H are

$$\begin{array}{lll} H = \{e, \rho^3\} & \rho H = H\rho = \{\rho, \rho^4\} & \rho^2 H = H\rho^2 = \{\rho^2, \rho^5\} \\ \tau H = H\tau = \{\tau, \rho^3\tau\} & \rho\tau H = H\rho\tau = \{\rho\tau, \rho^4\tau\} & \tau\rho^2 H = \rho^2\tau H = \{\rho^2\tau, \rho^5\tau\}. \end{array}$$

□

(d) $G \subset S_5$ where G is the set of all $\sigma \in S_5$ that stabilize $\{1,2,3\}$. This means

$$G = \Big\{\sigma \in S_5 \;\Big|\; \{\sigma(1), \sigma(2), \sigma(3)\} = \{1,2,3\}\Big\}.$$

Solution. G consists of the elements

$$e, (12), (13), (23), (123), (132), (45), (12)(45), (13)(45), (23)(45), (123)(45), (132)(45).$$

There will be 10 left cosets of G. They are:

$$\begin{aligned}
G &= \{g \in G | g(\{1,2,3\}) = \{1,2,3\}\} \\
(14)G &= \{g \in G | g(\{1,2,3\}) = \{2,3,4\}\} \\
(15)G &= \{g \in G | g(\{1,2,3\}) = \{2,4,5\}\} \\
(24)G &= \{g \in G | g(\{1,2,3\}) = \{1,3,4\}\} \\
(25)G &= \{g \in G | g(\{1,2,3\}) = \{1,3,5\}\} \\
(34)G &= \{g \in G | g(\{1,2,3\}) = \{1,2,4\}\} \\
(35)G &= \{g \in G | g(\{1,2,3\}) = \{1,2,5\}\} \\
(14)(25)G &= \{g \in G | g(\{1,2,3\}) = \{3,4,5\}\} \\
(14)(35)G &= \{g \in G | g(\{1,2,3\}) = \{2,4,5\}\} \\
(24)(35)G &= \{g \in G | g(\{1,2,3\}) = \{3,4,5\}\}.
\end{aligned}$$

There are also 10 right cosets of G:

$$\begin{aligned}
G &= \{g \in G | g^{-1}(\{1,2,3\}) = \{1,2,3\}\} \\
G(14) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{2,3,4\}\} \\
G(15) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{2,4,5\}\} \\
G(24) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{1,3,4\}\} \\
G(25) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{1,3,5\}\} \\
G(34) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{1,2,4\}\} \\
G(35) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{1,2,5\}\} \\
G(14)(25) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{3,4,5\}\} \\
G(14)(35) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{2,4,5\}\} \\
G(24)(35) &= \{g \in G | g^{-1}(\{1,2,3\}) = \{3,4,5\}\}.
\end{aligned}$$

$\square$

Exercise 2. Let A be a subgroup of B. The **index** of A in B is the number of *left*[1] cosets of A in B. Find the indices of each of the following subgroups: $\leftarrow_1$

(a) $\{e, (12)(34), (13)(24), (14)(23)\} \subset S_4$

Solution. The subgroup has 4 elements and S_4 has 24 elements so the index is $\frac{24}{4} = 6$. $\square$

(b) $\langle (23)(14657) \rangle \subset S_7$

[1] correction: the word left was omitted originally; thanks Dalton

Solution. The order of this element is 10 so the subgroup it generates has 10 elements. Therefore the index is $\frac{7!}{10} = 7 \cdot 6 \cdot 4 \cdot 3 = 504$. $\square$

(c) $G \subset S_7$ where G is isomorphic to $S_3 \times S_3$

Solution. The index of G in S_7 is $\frac{\#S_7}{\#G} = \frac{7!}{3! \cdot 3!} = 7 \cdot 5 \cdot 4 = 140$. $\square$

(d) $15\mathbf{Z}/60\mathbf{Z} \subset \mathbf{Z}/60\mathbf{Z}$

Solution.

$$\frac{\#(\mathbf{Z}/60\mathbf{Z})}{\#(15\mathbf{Z}/60\mathbf{Z})} = \frac{60}{4} = 15.$$

$\square$

Exercise 3. Let $\varphi : A \to B$ be a homomorphism with kernel K. For each $b \in B$, let $\varphi^{-1}(b) = \{a \in A \mid \varphi(a) = b\}$. Show that if $\varphi^{-1}(b)$ is not empty then it is both a left and right coset of K. (Hint: show that $\varphi^{-1}(b)$ is a left coset by picking any $a \in \varphi^{-1}(b)$ and checking that $\varphi^{-1}(b) = aK$.)

Solution. To prove that $\varphi^{-1}(b)$ is a left coset of K we have to show that $\varphi^{-1}(b) = aK$ for some $a \in A$. Since $\varphi^{-1}(b)$ is non-empty by assumption, there is some $a \in \varphi^{-1}(b)$. I claim that $\varphi^{-1}(b) = aK$. First of all, any element of aK can be written as ax for some $x \in K$ and $\varphi(ax) = \varphi(a)\varphi(x) = be = b$ since $\varphi(a) = b$ and $\varphi(x) = e$. This tells us that $aK \subset \varphi^{-1}(b)$.

Now we'll check that $\varphi^{-1}(b) \subset aK$. Suppose $a' \in \varphi^{-1}(b)$. Let $x = a^{-1}a'$. Then

$$\varphi(x) = \varphi(a^{-1}a') = \varphi(a)^{-1}\varphi(a') = b^{-1}b = e.$$

This means that $x \in K$, by definition of the kernel. So $a' = ax \in aK$. Thus $\varphi^{-1}(b) \subset aK$ so the two sets aK and $\varphi^{-1}(b)$ must be the same.

This proves that $\varphi^{-1}(b)$ is a left coset. We must also show it is a right coset. The proof is very similar so I'll do it a little faster this time: we have

$$\varphi(Ka) = \varphi(K)\varphi(a) = eb = b$$

so $Ka \subset \varphi^{-1}(b)$. On the other hand, if $a' \in \varphi^{-1}(b)$ then $\varphi(a') = b$ so $\varphi(a'a^{-1}) = e$. Therefore $a'a^{-1} \in K$ so $a' \in Ka$. This shows that $\varphi^{-1}(b) \subset Ka$ and completes the proof. $\square$

Exercise 4. Show that for $n \geq 2$, the set of odd permutations is a left and right coset of A_n in S_n.

Solution. The odd permutations are $\mathrm{sgn}^{-1}(-1)$ so by the previous exercise, they are a left and right coset of $\ker(\mathrm{sgn}) = A_n$. $\square$