Math 3140 — Fall 2012
Assignment #4

Due Fri., Sept. 28. Remember to cite your sources, including the people you talk to.

In this problem set, we use the notation $\gcd\{a, b\}$ for the greatest common divisor of a and b.

Exercise 27. [Fra, §6, #16]

1. [Arm, §5, #10]. Make a list of all elements of $\mathbf{Z}/12\mathbf{Z}$ that generate $\mathbf{Z}/12\mathbf{Z}$. Recall that an element g of a group G **generates** G if every element of G is of the form g^n for some integer n.

2. An **automorphism** of a group is an isomorphism from that group to itself. Find all automorphisms of the group $\mathbf{Z}/12\mathbf{Z}$. (Hint: if φ is an automorphism of $\mathbf{Z}/12\mathbf{Z}$, what values are possible for $\varphi(1)$?)

Exercise 30. Let X be a set and $Y \subset X$ a subset. Let G be the subset of all $g \in S_X$ such that $g(y) \in Y$ for all $y \in Y$. Show that *if the function $g|_Y : Y \to Y$ is surjective*[1] then G is a subgroup of S_X. $\leftarrow_1$

Exercise 31. [Fra, §8, #21]

(a) Verify that the six matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

form a group in which the operation is multiplication of matrices. (Hint: one solution to this problem uses the last exercise; let $X = \mathbf{R}^3$ and let $Y = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ are the standard basis vectors.)

(b) Show that this group is isomorphic to S_3.

Exercise 32. In the **fifteen puzzle** you are given a 4×4 grid containing 15 square tiles and one empty space. Any tile adjacent to the empty space can be slid into the empty space. The goal is, given a scrambled puzzle like the one on the left below

2	13	4	8
5	1	3	7
9	12	6	15
10	14	11	

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

[1]Correction! I left out this important hypothesis originally; thanks Tyler

to return it to the starting position on the right. If you haven't played with a fifteen puzzle before, you may want to practice online before doing this exercise: migo.sixbit.org/puzzles/fifteen/.

(a) Explain a correspondence between every position of the fifteen puzzle and the elements of the group S_{16}.

(b) Write down the permutation corresponding to the unsolved fifteen puzzle above, using cycle notation.

(c) Demonstrate that under this correspondence each move on a 15-puzzle corresponds is a transposition.

(d) Show that a sequence of moves beginning with the empty tile in the upper left and ending with it in the lower right must consist of an even number of moves.

(e) Write down the permutation correpsonding to the unsolved fifteen puzzle below:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

(f) Show that the fifteen puzzle shown above cannot be solved.

Exercise 33. (a) [Fra, §6, #23]. List all subgroups of $\mathbf{Z}/36\mathbf{Z}$ and indicate which subgroups are contained in which others.

(b) Describe all subgroups of $\mathbf{Z}/n\mathbf{Z}$.

Exercise 34. (a) Let n be an integer. Show that the order of $k \in \mathbf{Z}/n\mathbf{Z}$ is $n/\gcd\{k,n\}$.

(b) Let m and n be integers. Show that there is a homomorphism $\varphi : \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ such that $\varphi(1) = k$ if and only if $n/\gcd\{k,n\}$ divides m.

Exercise 35. Show that if G is an **abelian** group and n is a positive integer then

$$H = \left\{ g \in G \middle| g^n = 1 \right\}$$

is a subgroup of G. (Hint: One way to solve this exercise is to show that $\varphi(g) = g^n$ is an homomorphism from G to itself.)

References

[Arm] M. A. Armstrong. *Groups and symmetry*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1988.

[Fra] John B. Fraleigh. *A first course in abstract algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., seventh edition edition, 2002. ISBN-10: 0201763907, ISBN-13: 978-0201763904.