Math 3140 — Fall 2012
Assignment #3

Due Fri., Sept. 21. Remember to cite your sources, including the people you talk to.

Exercise 12. Suppose that G is a group with 5 elements.

(a) Show that G is isomorphic to $\mathbf{Z}/5\mathbf{Z}$, the group of integers modulo 5. (Hint: Let x be a non-zero element of G and consider the permutation L_x of G induced by left multiplication by x. What could the orbits of this action look like?)

(b) Suppose p is a prime number. Up to isomorphism, how many groups are there with p elements? (You do not have to prove your answer is correct, but try to give a sentence or two of justification.)

Exercise 21. Justify your answers below.

(a) Is $[0,1] \subset \mathbf{R}$ a subgroup? (Recall that $[0,1]$ is the interval of all real numbers x such that $0 \leq x \leq 1$.)

(b) Is $3\mathbf{Z} \subset \mathbf{R}$ a subgroup? (Here $3\mathbf{Z}$ is the set of integers that are multiples of 3.)

(c) Is $\mathbf{Q} \subset \mathbf{C}$ a subgroup?

(d) Is $\mathbf{R} \smallsetminus \mathbf{Q} \subset \mathbf{R}$ a subgroup? (Note that $\mathbf{R} \smallsetminus \mathbf{Q}$ is the set of all irrational real numbers.)

(e) Let $A \subset D_n$ be the subset consisting of all reflections and the identity. Is A a subgroup?

(f) Let $B \subset D_n$ be the subset of all rotations (including the identity). Is B a subgroup?

Exercise 22. Suppose that $\varphi : A \to B$ is a homomorphism of groups. Let $K \subset A$ be the set of all elements $a \in A$ such that $\varphi(a) = 1$. Symbolically,

$$K = \{ a \in A \mid \varphi(a) = 1 \}.$$

Show that K is a subgroup of A.

Exercise 23. [Arm, Exercise 5.1].

(a) Find all subgroups of $\mathbf{Z}/12\mathbf{Z}$.

(b) Find all subgroups of D_5.

Exercise 24. Suppose that G is a group and A and B are subgroups of G.

(a) Show that $A \cap B$ is also a subgroup.

(b) Assume that G is abelian. Let $C = \{ab \mid a \in A, b \in B\}$. Show that C is a subgroup of G.

(c) Show that in the last part, the assumption G be abelian is essential by giving an example of a non-abelian group G and subgroups A and B such that if C is defined as above then C is not a subgroup of G.

Exercise 25. Compute the sign of each of the following permutations:

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$

(b) $(1364)(25)$

(c) $(a_1 a_2 \cdots a_n)$.

Exercise 26. The first 3 parts of this problem are meant to give you ideas for the last part. It is also permissible to use the last part to solve the first 3 parts.

(a) Compute the order of the permutation $(12)(345) \in S_5$.

(b) Compute the order of $(123)(4567) \in S_7$.

(c) Compute the order of $(12)(34)(567) \in S_7$.

(d) Suppose that $\sigma = \sigma_1 \cdots \sigma_k$ is a product of **disjoint** cycles in S_n. Prove that

$$\operatorname{ord}(\sigma) = \operatorname{lcm}\{\operatorname{ord}(\sigma_1), \ldots, \operatorname{ord}(\sigma_k)\}.$$

Exercise 27. An **automorphism** of a group is an isomorphism from that group to itself. Find all automorphisms of the group $\mathbf{Z}/12\mathbf{Z}$.

Exercise 28. Let A be a group. Let be the set of automorphisms of A:

$$G = \{f : A \to A \mid f \text{ is an isomorphism of groups}\}.$$

Show that G is a group where the operation is composition of functions. (Hint: show that G is a subgroup of S_A.)

Exercise 29. Let G be the set of all **surjective** functions from $\mathbf{N}$ (the set of natural numbers) to itself. Is G a group with the operation being composition of functions? If so, prove it. If not, say which axioms of a group fail.

References

[Arm] M. A. Armstrong. *Groups and symmetry.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1988.