Math 3140 — Fall 2012
Assignment #2

Fri., Sept. 14[1] $\leftarrow_1$

Exercise 2. [Fra, §4, #9]. Let U be the set of complex numbers of absolute value 1.

(a) Show that U is a group under multiplication of complex numbers.

(b) Show that U is not isomorphic to $\mathbf{R}$ (with its additive group structure).

(c) Show that U is not isomorphic to $\mathbf{R}^*$ (with its multiplicative group structure).

Hint: it might help to think about Exercise 13 while thinking about this one.

Exercise 13. Suppose that G is a group. An element $x \in G$ is said to have **order** n if $x^n = e$ but $x^k \neq e$ for $0 < k < n$. If no such n exists, we say that x has infinite order.

(a) What is the order of the identity element e?[2] $\leftarrow_2$

(b) Compute the orders of all of the elements of the symmetric group S_3.

(c) Give an example of a group with more than one element where every element other than e has infinite order.

(d) Give an example of an infinite group where every element has finite order. (Hint: look for a subgroup of U.)

Exercise 14. Prove that S_3 is not isomorphic to $\mathbf{Z}/6\mathbf{Z}$, the group of integers modulo 6.[3] $\leftarrow_3$

Exercise 15. Suppose that G is a group and x is an element of G. Prove that the function $\varphi : G \to G$ defined by

$$\varphi(y) = xyx^{-1}$$

is an isomorphism from G to itself. An isomorphism from a group to itself is called an **automorphism**.

[1] Due date postponed!
[2] clarification added; thanks Rachel Benefiel
[3] this is the set $\{0, 1, 2, 3, 4, 5\}$ with addition modulo six as the group operation

Exercise 16. **Do not** show your work on this problem. Consider the following permutations in S_6:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$$
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$
$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix}$$

(a) [Fra, §8, #2]. Compute $\tau^2\sigma$.

(b) [Fra, §8, #8]. Compute σ^{100}.

(c) Express μ in cycle notation.

Exercise 17. Let σ be the permutaiton $(a_1 a_2 \cdots a_n)$ in cycle notation.

(a) Write σ^{-1} in cycle notation.

(b) Write σ^2 in cycle notation. Hint: your answer may depend on n; try computing σ^2 for a few small values of n.

Exercise 18. Let A_4 be the subset of S_4 consisting of all permutations that are products of an **even number**[4] of transpositions. A transposition is a permutation that exchanges two things and leaves everything else stationary; in cycle notation, a transposition looks like (ab). Thus $(12)(13)$ and $(12)(24)(13)(24)$ are in A_4, but (23) and $(12)(23)(34)$ are not. ←4

Write down all of the elements of A_4 using cycle notation.[5] ←5

Exercise 19. [Fra, §8, #18]. List all of the subgroups of S_3.

Exercise 20. Suppose that G is a group. How would you define a symmetry of G?

References

[Fra] John B. Fraleigh. *A first course in abstract algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., seventh edition edition, 2002. ISBN-10: 0201763907, ISBN-13: 978-0201763904.

[4] correction: accidentally left this out before!

[5] I reworded this problem; note that you **do not** have to write down the multiplication table!