Math 3140 — Fall 2012
Assignment #1

Due Weds., Sep. 5.

Exercise 1. (a) List all of the symmetries of a square, allowing all transformations made up of rotations, reflections, and translations. This group is called D_4.

(b) How many elements does D_4 have?

(c) Is D_4 abelian?

Exercise 3. (a) Suppose that G is a group and x is an element of G. Show that if y is a right inverse of x (meaning that $xy = e$) and z is a left inverse of x (meaning that $zx = e$) then $y = z$.

(b) Conclude from this that each element of G has exactly one inverse. (Remember that the group axioms explicity guarantee every element has *at least one* inverse, but they do not say that the inverse must be unique. You are supposed to prove this uniqueness.)

Exercise 4. Suppose that G is a group and x and y are elements of G. Show that $(xy)^{-1} = y^{-1}x^{-1}$.

Exercise 5. [Fra, §4, #19]. Let S be the set of all real numbers except -1 with the composition law

$$a * b = a + b + ab.$$

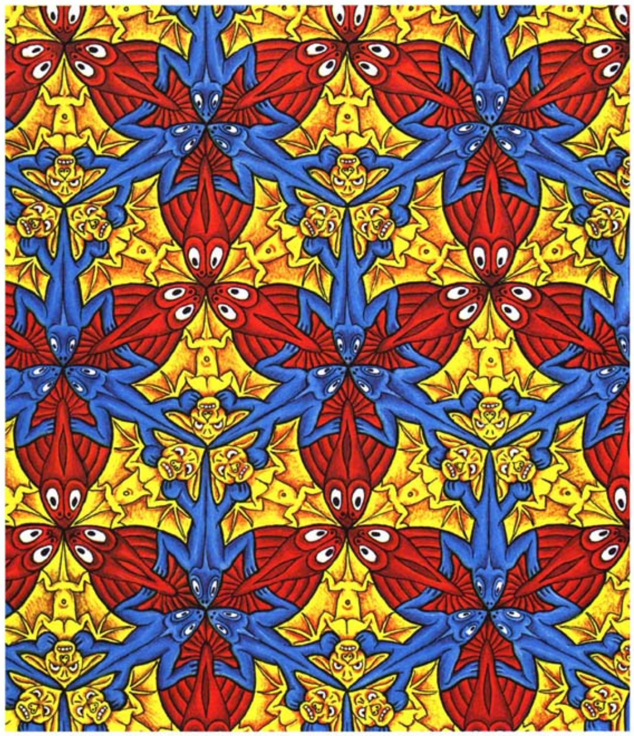
(a) Show that S is a group.

(b) Show that S is isomorphic to $\mathbf{R}^*$.

(c) Solve the equation $2 * x * 3 = 7$ in S. (Suggestion: While it is possible to solve this problem directly using the definition of the group law, try making use of the previous part of this exercise.)

Exercise 6. [Fra, §4, #35]. Show that if G is a group containing elements a and b and $(ab)^2 = a^2b^2$ then $ab = ba$.

Exercise 7. Describe all of the symmetries of the picture below (assume that it repeats to infinity in all directions):



M. C. Escher, Tessellation 85.
Source: http://britton.disted.camosun.bc.ca/jbsymteslk.htm

Exercise 8. How many symmetries does a set with n elements have? (How many permutations are there of a set with n elements?)

Exercise 9. Consider the following six functions:

$$f_1(x) = x \qquad f_2(x) = 1 - x \qquad f_3(x) = \frac{1}{x}$$
$$f_4(x) = \frac{1}{1-x} \qquad f_5(x) = \frac{x}{x-1} \qquad f_6(x) = \frac{x-1}{x}.$$

(a) Show that these functions form a group where the group operation is *composition of functions*.

(b) Show that this group is isomorphic to S_3 by constructing an isomorphism.

Exercise 10. Let S_4 be the group of symmetries of a set with 4 elements. Let G be the group of rigid symmetries (compositions of translations, rotations, and reflections are allowed) of a regular tetrahedron. Show that G and S_4 are isomorphic. (Hint: there is a more efficient way to do this than writing down both multiplication tables!)

Exercise 11. Suppose that A and B are groups and $f : A \to B$ is a function that satisfies the property

$$f(xy) = f(x)f(y).$$

(Functions of this type are called **homomorphisms** and will be very important later.)

(a) Show that $f(e) = e$.

(b) Show that if x is in A then $f(x^{-1}) = f(x)^{-1}$.

References

[Fra] John B. Fraleigh. *A first course in abstract algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., seventh edition edition, 2002. ISBN-10: 0201763907, ISBN-13: 978-0201763904.