Math 3140 — Fall 2012
Handout #14

Exercise 1. (a) Is the ring $\mathbf{F}_2[x]/(x^2+1)$ a field?

(b) Is the ring $\mathbf{F}_2[x]/(x^2+x)$ a field?

Exercise 2. Let A be the ring $\mathbf{F}_2[x]/(x^2+x+1)$.

(a) How many elements does A have? List them.

(b) Which elements of A are invertible?

Exercise 3. Let B be the ring $\mathbf{F}_2[x]/(x^3+x+1)$.

(a) How many elements does B have? List them.

(b) Which elements of B are invertible?

Exercise 4. Let C be the ring $\mathbf{F}_3[x]/(x^2+1)$.

(a) How many elements does C have? List them.

(b) Which elements of C are invertible?

(c) Is C isomorphic to $\mathbf{F}_3 \times \mathbf{F}_3$?

Exercise 5. Let D be the ring $\mathbf{F}_5[x]/(x^2+1)$.

(a) Find all of the elements y in $\mathbf{F}_5 \times \mathbf{F}_5$ such that $y^2+1=0$.

(b) Use your list to construct an isomorphism $\mathbf{F}_5[x]/(x^2+1) \cong \mathbf{F}_5 \times \mathbf{F}_5$.

Exercise 6. (a) Make a list of prime numbers p for which $\mathbf{F}_p[x]/(x^2+1) \cong \mathbf{F}_p \times \mathbf{F}_p$.

(b) Make a list of prime numbers p for which $\mathbf{F}_p[x]/(x^2+1)$ is a field.

(c) Are there any prime numbers for which neither of the above occurs?

(d) Make a conjecture about the general pattern.

Exercise 7. Let

$$A = \left\{ \begin{pmatrix} a & a+b \\ b & a \end{pmatrix} \;\middle|\; a, b \in \mathbf{F}_2 \right\}.$$

(a) How many elements does A have? List them.

(b) Verify that if X and Y are elements of A then XY is an element of A.

(c) (skip in class) Verify that A is a ring.

(d) Which elements of A are invertible?

Exercise 8. Which of the following rings are isomorphic to each other?

(i) the ring A from the previous exercise

(ii) $\mathbf{F}_2[x]/(x^2+1)$

(iii) $\mathbf{F}_2[x]/(x^2)$

(iv) $\mathbf{F}_2 \times \mathbf{F}_2$

(v) $\mathbf{F}_2[x]/(x^2+x)$

(vi) $\mathbf{F}_2[x]/(x^2+x+1)$

Exercise 9. How many non-isomorphic commutative rings can you get with the construction

$$\mathbf{F}_3[x]/(x^2+ax+b)$$

where $a, b \in \mathbf{F}_3$?

Exercise 10. Can you construct a field with

(a) 10 elements?

(b) 15 elements?

(c) 16 elements?

(d) 21 elements?

(e) 25 elements?

(f) 27 elements?

(g) 12 elements?

(h) 18 elements?