Math 3140 — Fall 2012
Handout #11

Exercise 1. An element of a ring R is called a **unit** if it has a multiplicative inverse. The set of all units of R is denoted R^*.

(a) Let $\mathbf{R}[x]$ be the ring of polynomials with real coefficients. What is $\mathbf{R}[x]^*$?

(b) Let $R = \mathbf{R}[i]/(i^2 = -1) = \mathbf{C}$. What is R^*?

(c) Let $R = \mathbf{R}[\epsilon]/(\epsilon^2 = 0)$ be the ring of symbols $a + b\epsilon$ with

$$(a + b\epsilon) + (a' + b'\epsilon) + (a + a') + (b + b')\epsilon$$
$$(a + b\epsilon)(a' + b'\epsilon) = aa' + (ab' + a'b)\epsilon.$$

What is R^*?

(d) Let $R = \mathbf{R}[x]/(x^2 = 1)$ be the ring of symbols $a + bx$ with operations

$$(a + bx) + (a' + b'x) = (a + a') + (b + b')x(a + bx)(a' + b'x) = (aa' + bb') + (ab' + a'b)x.$$

What is R^*?

The ring of **Gaussian integers** is the set of all complex numbers $x + iy$ where $x, y \in \mathbf{Z}$. The Gaussian integers are denoted $\mathbf{Z}[i]$. The addition and multiplication rules are just as in the complex numbers:

$$(x + iy) + (x' + iy') = (x + x') + i(y + y')$$
$$(x + iy)(x' + iy') = xx' - yy' + i(xy' + x'y).$$

Exercise 2. An element of a ring is called a **unit** if it has a multiplicative inverse.

(a) List of all units of $\mathbf{Z}[i]$.

(b) Can you prove that your list is complete?

Exercise 3. An element x of a commutative ring R is called **irreducible** if it cannot be written as yz unless y or z is a unit.

(a) What are the irreducible elements of $\mathbf{Z}$?

(b) Is $2 = 2 + 0i$ irreducible in $\mathbf{Z}[i]$?

(c) What about 3?

(d) Make a list of some irreducible elements of $\mathbf{Z}$ and determine which are irreducible in $\mathbf{Z}[i]$.

(e) Make a conjecture about which irreducible elements of $\mathbf{Z}$ remain irreducible in $\mathbf{Z}[i]$.

Exercise 4. Hint: it might be helpful to think about both parts of this exercise at the same time.

(a) Let $\mathbf{C}[x]$ be the ring of polynomials with complex coefficients. What are the irreducible elements of $\mathbf{C}[x]$?

(b) Let $\mathbf{R}[x]$ be the ring of polynomials with real coefficients. What are the irreducible elements of $\mathbf{R}[x]$?

The ring of **Eisenstein integers** is the set of all complex numbers $x + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)y$ where $x, y \in \mathbf{Z}$. It is sometimes denoted $\mathbf{Z}[\rho]$ where $\rho = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. However, the notation $\mathbf{Z}[\rho]$ is much less standard than $\mathbf{Z}[i]$; the name "Eisenstein integers" is also not used as widely as "Gaussian integers" is. Addition and multiplication in the Eisenstein integers are again as in the complex numbers.

Exercise 5. Let $\rho = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ as above.

(a) Remember that we really should check that $\mathbf{Z}[\rho]$ is a ring. We won't do this now because we'll be able to do it more efficiently later.

(b) Compute ρ^2 and ρ^3 and verify that these are Eisenstein integers.

(c) Make a list of all the units in $\mathbf{Z}[\rho]$.

(d) Which irreducible elements of $\mathbf{Z}$ remain irreducible in $\mathbf{Z}[\rho]$?